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AN ALGORITHM TO COMPUTE A RULE FOR DIVISION PROBLEMS WITH MULTIPLE REFERENCES

FRANCISCA J. SÁNCHEZ SÁNCHEZ
fsansan@upo.es
Pablo de Olavide University
Departamento de Economía, Métodos Cuantitativos e Historia Económica
Carretera de Utrera, Km. 1, 41013 Seville (Spain)

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ABSTRACT: In this paper we consider an extension of the classic division problem with claims: The division problem with multiple references. Hinojosa et al. (2012) provide a solution for this type of problems. The aim of this work is to extend their results by proposing an algorithm that calculates allocations based on these results. All computational details are provided in the paper.

Keywords: Division Problems, Multiple References, Cooperative Games, Talmud Rule, Algorithm.

1. Introduction

This paper is focused on the division problems with claims. A division problem consists of dividing a determined quantity among a group of agents according to certain "characteristics" or "references" related to these agents.

The simplest case of these problems, and the most studied in the literature, is that in which each agent is characterized by a single number or a single parameter which is the reference or characteristic of the agent. This model is defined as "the classic division problem". Such situations are presented in a variety of real problems. Classic examples appear in the Babylonian Talmud (see Aumann and Maschler, 1985) in a context of bankruptcy, where the amount to divide (this amount is called the *estate*) is insufficient to satisfy the claims of creditors.

The classic division problem can be extended in order to represent and analyze more situations in modern life in a more realistic way. The classic division problem is presented in which the reference of each agent is multi-dimensional, and therefore, the agents involved in the problem are now characterized by several parameters, instead of just one. In this paper these are referred to as multiple references.

The following situation can be represented as a division model with multiple references: A certain project can be carried out in several countries, but there is no certainty as to which country will finally develop such a project. There are several multinational enterprises involved at different levels in the development of the project. These enterprises have perfectly calculated their costs of participation according to the candidate country where the project is developed. These costs will vary depending on the country where the project is carried out, because the legal aspects are vary per country and the infrastructures of the multinational enterprises also differ. These costs determine the references. The project receives financial support from an international organism, and the problem is to divide the costs among the participant enterprises.

The Talmud rule is one of the most prominent rules in classic division problems. This rule can be generalized to the division problem with multiple references. Hinojosa et al. (2012) used the Theory of Cooperative Games in order to provide an extension of the Talmud rule in this more general context. They define a single-valued rule for these problems, the nucleolus of the coalitional game, which assigns the maximum value attained across the various references to each coalition.

An important aspect to consider is the computational complexity of solution concepts, based on the game theory. In a cooperative game with n-players, the number of valuations of the characteristic function of the game grows exponentially with the number of players. Therefore any algorithm for the computation of solutions, such as the core, the Shapley value or the nucleolus, which requires the manipulation of data, will have to take this fact into account. In recent years, the interest in computational complexity aspects has been growing (see, for example, Deng and Papadimitriou (1994), Deng et al. (1997), or Granot et al. (1998)).

Our interest is focused on the computation of the nucleolus of the game proposed by Hinojosa et al. (2012). Unfortunately, there exists no closed-form formula for the nucleolus solution; in the literature the usual practice is to compute it an iterative manner by solving a series of linear programs (see, for example, Maschler et al. (1979), Owen (1995)). Our proposal consists of an algorithm which calculates the allocations that the nucleolus provides. This procedure also shows the complete path of awards of the rule.

2. The classic division problem

An estate $E \in \mathbb{R}_+$ of an infinitely divisible resource, has to be divided between $N = \{1, \ldots, n\}$ agents according to certain references, represented by the vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, where $c_i \in \mathbb{R}_+$ represents the reference or characteristic of the agent $i \in N$. A classic division problem is represented by $(N, \mathbf{c}, E)$. Let $\mathcal{C}^N$ be the class of these problems. When there are no possible confusions the division problem is denoted as $(\mathbf{c}, E)$.

The same formal model arises in a variety of contexts, for example, inheritance problems (O'Neill 1982), in taxation problems (Young 1988, 1990), in bankruptcy problems (Aumann and Maschler, 1985) and in cost-sharing problems (Moulin, 1987).

The purpose of this model is to determine the amounts that the agents will receive so that the total sum is the total amount to be distributed. Formally, an allocation for a problem $(\mathbf{c}, E) \in \mathcal{C}^N$ is a vector $\mathbf{x} \in \mathbb{R}^N_+$, which satisfies the efficiency requirement, $\sum_{i \in N} x_i = E$. Each component, x_i , represents the assigned amount for the agent i in the division problem. Let $X(E) \subseteq \mathbb{R}^N_+$ be the set of all the allocations of the estate E. A *division rule* is a function R, that associates an allocation $R(\mathbf{c}, E) \in X(E)$ with each division problem $(\mathbf{c}, E) \in \mathcal{C}^N$. A good revision of the classic division rules can be seen in Thomson (2003).

One division rule that has been widely studied, in the case of bankruptcy problems, is the Talmud rule (Aumann and Maschler, 1985). Suppose that $c_1 \leq c_2 \leq \ldots \leq c_n$ are the claims for an estate E and that $\sum_{i=1}^n c_i \geq E$. For each division problem $(\mathbf{c}, E) \in \mathcal{C}^N$, the Talmud rule assigns to each agent $i \in N$,

$$T_i(\mathbf{c}, E) = \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } E \leq \sum_{i \in N} \frac{c_i}{2} \\ c_i - \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is such that $\sum_{i \in N} T_i(\mathbf{c}, E) = E$ is satisfied.

When the agents adopt cooperation agreements, a valid tool to study the division problems is the Theory of Cooperative Games. A cooperative game is represented by a pair (N, v) where $N = \{1, 2, \ldots, n\}$ is the set of agents or players, v is a function defined in the set of coalitions (subset of N), with values in $\mathbb{R}$, and $v(\emptyset) = 0$. The value $v(S)$, $S \subseteq N$, is a measure of what a coalition can obtain by itself, without the cooperation of the players in $N \setminus S$. In a cooperative game, (N, v), it is often assumed that all the players decide to work together, whereby a *grand coalition* N is formed. The central problem is to find a "fair" distribution of the total value $v(N)$ among the individual players $i \in N$.

A classic division problem can be associated with a cooperative game in different ways. The most widely used way to do it, given in the literature was introduced by O'Neill in 1982. O'Neill associates a cooperative game with the division problem $(\mathbf{c}, E) \in \mathcal{C}_N$, denoted by $(N, v_{(\mathbf{c},E)})$. This game is defined as follows:

$$v_{(\mathbf{c},E)}(S) = \max\left\{E - \sum_{i \notin S} c_i, 0\right\}, \ \forall\, S \subset N.$$

The value $v_{(\mathbf{c},E)}(S)$ is a pessimistic valuation of what the coalition S can arrive at, since the first values assigned to the agents which do not belong to S, determine its reference and if an amount remains to be divided, E, it would be the amount that coalition S could have guaranteed. Note that, in O'Neill's game, the value of the grand coalition coincides with the total amount to divide E ($v_{(\mathbf{c},E)}(N) = E$). In what follows, v will be denoted instead of $v_{(\mathbf{c},E)}$, to simplify presentation.

An allocation for this game is a vector, $\mathbf{x} \in \mathbb{R}^n_+$, such that $\sum_{i \in N} x_i = v(N)$, where x_i represents the awards or the payoff to player i. The sum $x(S) = \sum_{i \in S} x_i$ is the payoff of the coalition S. $I^*(N, v)$ is the set of allocations of the game.

A solution concept for cooperative games is a correspondence which associates each game with a nonempty set of allocations of the game. In this paper the nucleolus is used (Schmeidler, 1969) as a concept solution. A good revision can be found in Maschler (1992).

The nucleolus of the game (N, v) is defined as:

$$\begin{aligned} N(N, v) = \{\mathbf{x} \in I^*(N, v) | H_{2^n-2}(e(\mathbf{x}, S_1), e(\mathbf{x}, S_2), \ldots, e(\mathbf{x}, S_{2^n-2})) \leq_L \\ \leq_L H_{2^n-2}(e(\mathbf{y}, S_1), e(\mathbf{y}, S_2), \ldots, e(\mathbf{y}, S_{2^n-2})), \forall\, \mathbf{y} \in I^*(N, v)\}, \end{aligned}$$

where $e(\mathbf{x}, S) = v(S) - x(S)$ measures the dissatisfaction of coalition S at $\mathbf{x}$. $H_{2^n-2} : \mathbb{R}^{2^n-2} \longrightarrow \mathbb{R}^{2^n-2}$ is a correspondence which orders vectors of dimension $2^n - 2$ into decreasing order and $\leq_L$ means "non-greater" with respect to lexicographical order. The convexity and compacity of the set $I^*(N, v)$, ensures that the nucleolus is a unique allocation (Owen, 1995).

In a bankruptcy context, the Talmud rule coincides with the nucleolus of the corresponding cooperative game (Aumann and Maschler, 1985). Serrano (1995) shows that in surplus sharing problems, when $D = \sum_{i=1}^n c_i < E$, the nucleolus assigns each player $i \in N$ the amount $x_i = c_i + \frac{E-D}{n}$, thereby extending the Talmud rule, in these problems, which implies an equal split of the contested amount, which in this case is the surplus.

In this paper these ideas are generalized by considering an extension of the classic division problem to situations in which the characteristic of each agent is multidimensional, and therefore, several vectors of references have to be taken into account in the division.

3. The division problem with multiple references

Consider a fixed finite set of issues $M = \{1, 2, \ldots, m\}$, a *division problem with multiple references*, is a terna (N, C, E) where $N = \{1, 2, \ldots, n\}$ is the set of agents, $E \in \mathbb{R}{+}{+}$ is the estate to be allocated, and $C \in \mathbb{R}^{N \times M}_+$ is the *matrix of references.* [1]

[1] $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$) denotes the set of all (non-negative, positive) real numbers, and $\mathbb{R}^N$ ($\mathbb{R}^N_+$, $\mathbb{R}^N_{++}$) the Cartesian product of $|N|$ copies of $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$), where $|N|$ denotes the cardinal of N.

An element of matrix C is denoted by c_i^j. For each $i \in N$, the ith row of matrix C, $\mathbf{c}_i \in \mathbb{R}^M$, represents the references of agent i with respect to the different issues. For each $j \in M$, the column $\mathbf{c}^j \in \mathbb{R}^N$ represents the references of all the agents corresponding to the jth issue. Matrix C is also denoted as $(\mathbf{c}_i)_{i\in N}$ or as $(\mathbf{c}^j)^{j\in M}$. When there is no possible confusion, the division problem with multiple references is denoted as (C, E).

The class of all division problems with multiple references associated with the set of agents N and the set of issues M is denoted by $\mathcal{D}_N^M$.

An *allocation* for $(C, E) \in \mathcal{D}_N^M$ is a vector $\mathbf{x} \in \mathbb{R}_+^N$, which satisfies the efficiency requirement, $\sum_{i\in N} x_i = E$. Let $X(E) \subseteq \mathbb{R}_+^N$ be the set of all the allocations for the division problem with multiple references, $(C, E) \in \mathcal{D}_N^M$.

A *division rule* over $\mathcal{D}_N^M$ is a function, R, that associates a unique allocation $R(C, E) \in X(E)$ with each problem $(C, E) \in \mathcal{D}_N^M$.

In Hinojosa et al. (2012) a rule was proposed which provides a unique allocation for division problems with multiple references, and which coincides with the nucleolus of the cooperative game and considers the maximum surplus across the issues as a measure of the dissatisfaction. In Section 4 a procedure for the implementation of this rule is presented.

3.1. *Talmud rule with multiple references*

For each division problem with multiple references, $(C, E) \in \mathcal{D}_N^M$, $|M|$ coalitional games, $(N, v^j_{(C,E)})$, $j \in M$, can be defined by the procedure proposed by ONeill (1982). That is, for each $j \in M$ and for each $S \subseteq N$, then $v^j_{(C,E)}(S) = \max\left\{E - c^j(N \setminus S), 0\right\}$, where $c^j(N \setminus S) = \sum_{i\in N\setminus S} c_i^j$.

The division rule is based upon the differences between what the coalitions obtain with a certain allocation and their values in the coalitional games defined above.

For each allocation, $\mathbf{x} \in X(E)$, and each coalition, $S \subseteq N$, the $|M|$ *surplus functions* are $e_{v^j_{(C,E)}}(\mathbf{x}, S) = v^j_{(C,E)}(S) - x(S)$, $j \in M$. These functions measure the dissatisfaction of coalition S at $\mathbf{x}$ with respect to all the issues, and plays a central role in the definition of the division rule.

The goal is to select allocations that are better in a lexicographic sense. If a unique vector of references is considered, then a lexicographical order among the allocations can be defined, and a unique best outcome can be determined, the nucleolus. For the case of several vectors of references, the maximum surplus across the issues is considered as a measure of the dissatisfaction of coalition S at $\mathbf{x} \in X(E)$, that is,

$$e_{(C,E)}(\mathbf{x}, S) = \max_{j\in M} \{e_{v^j_{(C,E)}}(\mathbf{x}, S)\} = \max_{j\in M} \{v^j_{(C,E)}(S)\} - x(S).$$

For each $\mathbf{x} \in X(E)$, a $(2^N - 2)$-dimensional vector, $\pi_{(C,E)}(\mathbf{x})$, is constructed with the maximum surplus, $e_{(C,E)}(\mathbf{x}, S)$, $S \subset N$, arranged in decreasing order. Vector $\pi_{(C,E)}(\mathbf{x})$ is a vector-valued measure of the performance of allocation $\mathbf{x}$ with respect to all the coalitions which take into account all the issues.

It is said that vector $\pi(\mathbf{x})$ is lexicographically better than vector $\pi(\mathbf{y})$, and it is denoted as $\pi(\mathbf{x}) <_{lex} \pi(\mathbf{y})$, if $\pi^k(\mathbf{x}) < \pi^k(\mathbf{y})$ for the first component, k, in which vector $\pi(\mathbf{x})$ and vector $\pi(\mathbf{y})$ are different. This binary relation defines a complete order and therefore, a division rule can be defined in the class $\mathcal{D}_N^M$ by selecting, for each $(C, E) \in \mathcal{D}_N^M$, the allocation which lexicographically minimizes $\pi(\mathbf{x})$ from among all the allocations $\mathbf{x} \in X(E)$.

Definition 1 *For each division problem with multiple references, $(C, E) \in \mathcal{D}_N^M$, the multiple-reference Talmudic rule, MT, is defined as $MT(C, E) = \text{arg lex-min}_{\mathbf{x}\in X(E)} \{\pi_{(C,E)}(\mathbf{x})\}$, where $\pi_{(C,E)}(\mathbf{x})$ is a vector whose components are the maximum surplus across issues, $e_{(C,E)}(\mathbf{x}, S)$, $S \subset N$, arranged in decreasing order.*

Note that for each $(C, E) \in \mathcal{D}_N^M$, the outcomes provided by the MT rule coincide with those obtained with the nucleolus of the coalitional game $(N, v_{(C,E)}^{\max})$, where $v_{(C,E)}^{\max}(S) = \text{máx}_{j \in M} \{v_{(C,E)}^{j}(S)\}$. This result follows from the fact that $e_{(C,E)}(\mathbf{x}, S) = e_{v_{(C,E)}^{\max}}(\mathbf{x}, S)$ for each $\mathbf{x} \in X(E)$ and each $S \subset N$.

In what follows, $v^{\max}$ is denoted instead of $v_{(C,E)}^{\max}$, and $e(\mathbf{x}, S)$ instead of $e_{v_{(C,E)}^{\max}}(\mathbf{x}, S)$ to simplify presentation.

3.2. *The path of awards of the rule*

For each $(C, E) \in \mathcal{D}_N^M$, denote by $\underline{\mathbf{c}}$ the vector whose components represent the minimum value from among the references of each agent, $\underline{c}_i = \min_{j \in M}\{c_i^j\}$, $i \in N$, and by $\underline{c}(N) = \sum_{i=1}^{n} \underline{c}_i$. Without any loss of generality, it is assumed that $N = \{1, 2, \ldots, n\}$ and $\underline{c}_1 \leq \underline{c}_2 \leq \ldots \leq \underline{c}_n$.

In Hinojosa et al. (2012) the path of awards[2] of the rule is analyzed, by studying two possible cases:

(a) The value of the estate is below the sum of the minimum references.

When the value of the estate does not exceed $\underline{c}(N)$, i. e., $E \leq \underline{c}(N)$, then the MT rule provides the same vector of awards as the Talmud rule for a classic division problem with references equal to $\underline{\mathbf{c}}$.

(b) The value of the estate is above the sum of the minimum references.

If $|N| = 2$ and $E > \underline{c}(N)$ when applying the MT rule, then the difference $E - \underline{c}(N)$ is allocated equally between the agents. As a consequence, for the two-agent case, the outcomes obtained with this rule coincide with those obtained with the classic Talmud rule with claims $\underline{\mathbf{c}}$. In Fig. 1. an example with two agents and two issues can be observed. In case a), the path of awards of the MT rule is presented, when $E \leq \underline{c}(N)$, and b), the complete path of awards of MT is given.

When $|N| \geq 3$ and $E > \underline{c}(N)$, then the path of awards of MT is piecewise linear, whereby the last path is piecewise parallel to line $x_1 = x_2 = \ldots = x_n$. In other words, for each matrix of references, the surplus above $\underline{c}(N)$ is distributed since proportionalities, which continue changing up to a certain value for E from which any additional amount is assigned in the same proportion among the agents.

4. Computational results

The computational complexity aspects of solution concepts in cooperative game theory have been widely studied in the literature. In our case, our interest is focused on the nucleolus. On one hand, some efficient[3] algorithms have been developed for the implementation of the nucleolus in some kinds of games, for instance in assignment games (Solymosi and Raghavan (1994)), in tree games (Meggido, 1978), and in convex games (4) (Kuipers, 1996). On the other hand it has been proved

[2]The path of awards of the rule are all allocations that provides a rule for different amounts to be divided, E, and for the reference vector, $\mathbf{c}$.

[3]The *size of a problem* is the length of the encoding, i.e., the number of *bits* necessary to represent it. The *running time* of an algorithm, $t(\xi)$, is defined as the maximum (computation) time required to solve any problem with size ξ. An algorithm is said to be *polynomial* or *efficient* if its running time $t(\xi)$ is bounded by a polynomial in ξ.

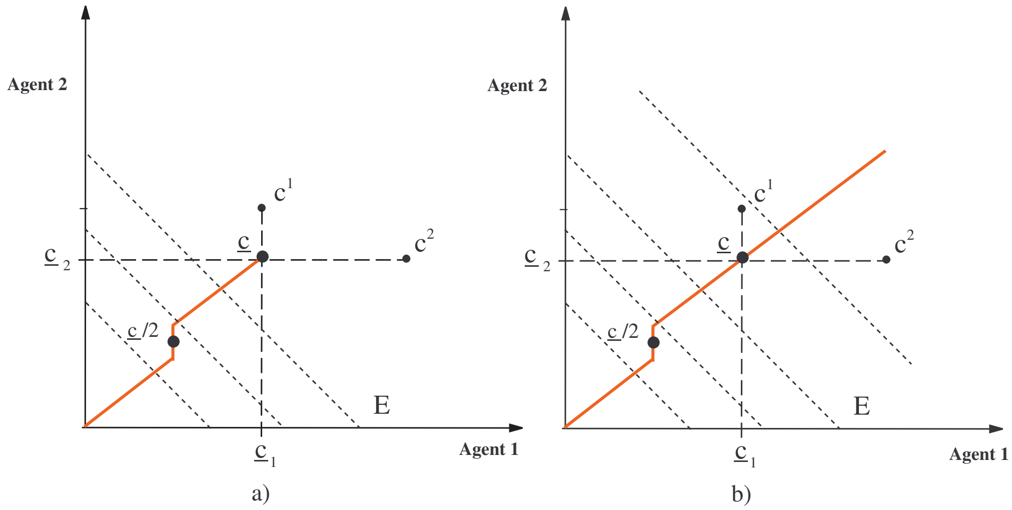


Fig. 1. The path of awards of MT.

that some problems are NP-hard[4], for instance, for testing core membership[5] (Faigle et al. 1997) or computing the nucleolus of minimum cost spanning tree games (Faigle et al. 1998).

Our goal is the computation of the nucleolus of the game $(N, v^{\max})$, which is not a convex game, as can be seen in the example shown below.

Example 1 *Consider the division problem with multiple references with three agents, $N = \{1, 2, 3\}$ and two issues. The references are $\mathbf{c}^1 = (3, 7, 10)^t$ and $\mathbf{c}^2 = (15, 9, 2)^t$. If we consider $E = 9$, then the game $(N, v^{\max})$ is the following:*

Table 1. The Game $(N, v^{\max})$.

S	$v^1(S)$	$v^2(S)$	$v^{\max}(S)$	S	$v^1(S)$	$v^2(S)$	$v^{\max}(S)$
{1}	0	0	0	{1,2}	0	7	7
{2}	0	0	0	{1,3}	2	0	2
{3}	0	0	0	{2,3}	6	0	6
				{1,2,3}	9	9	9

If $S = \{1, 2\}$ and $T = \{2, 3\}$,then the inequality $v^{\max}(S) + v^{\max}(T) \leq v^{\max}(S \cup T) + v^{\max}(S \cap T)$ is not satisfied, and the game is not convex.

The fact that game $(N, v^{\max})$ is not convex, complicates the computation of the allocation MT rule using an efficient algorithm. In general, no known algorithm computes the nucleolus either in polynomial time or efficiently (see Potter et al. (1996)).

Kohlberg (1972) shows that it is possible to obtain the nucleolus by solving a single linear program. However, this presents a very high number of constraints. Owen (1974) shows that this program can be reduced to another more manageable size. On the other hand, in Maschler et al. (1979) the nucleolus is obtained by the definition of the lexicographic centre of a cooperative game, that is iteratively solvedd as a series of linear programs. More recently, Sankaran (1991) reduced this sequence of problems. In Leng and

[4] NP (non polynomial) problems are those for which no known algorithm is developed in polymonial time. A problem is NP-hard when it can be demonstrated that it can develop in polynomial time, therefore all the NP problems would also be polymonial.

[5] The core of the game (N, v) is defined as $\{\mathbf{x} \in \mathbb{R}^N_+ | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N\}$.

Parlar (2010) an analytical method is proposed for computing the nucleolus without iterative calculations involving linear programs, although this is only valid for a game with a small number of players (for $n = 3$). In Puerto (2011), in a similar way to Owen (1974), a method is presented which permits the implementation of the nucleolus by solving a single linear program. The advantage of this method is that constraint coefficients are -1, 0, 1.

As noted above, it is usual practice to compute the nucleolus by solving iteratively a series of linear programs. In our case, this procedure has been adapted to obtain allocations which provide the MT rule in a division problem with multiple references. The algorithm provides the nucleolus of game $(N, v^{\max})$.

The maximum surplus, $v^{\max}(S) - x(S)$, of the different coalitions of agents with allocation $\mathbf{x}$ is denoted as ϵ. This ϵ, will be the value to be minimized in the problem. Given that $v^{\max}(S) = \max_{j \in M} \{v^j(S)\}$, with $v^j(S) = \max \left\{E - c^j(N \setminus S),\, 0\right\}$, $j \in M$, then in order to assign $v^j(S)$ as 0, when amount $E - c^j(N \setminus S)$ is not positive, two binary variables a_S^j and b_S^j, $\forall\, S \subset N$, $\forall\, j \in M$, are introduced into the algorithms, which take value 0 or 1 depending on the case. A larger positive constant is denoted as K.

Given the division problem with multiple references $(C, E) \in \mathcal{C}_N^M$, then the algorithm to compute the MT rule, is as follows:

Algorithm

Step 1: Solve the problem (P.1):

$$
\begin{aligned}
&\min \epsilon \\
&s.t: (E - c^j(N \backslash S))a_S^j - x(S) \le \epsilon,\ \forall\, S \subset N,\ \forall\, j \in M \\
&\qquad (E - c^j(N \backslash S)) - a_S^j K < 0,\ \forall\, S \subset N,\ \forall\, j \in M \\
&\qquad (E - c^j(N \backslash S)) + b_S^j K \ge 0,\ \forall\, S \subset N,\ \forall\, j \in M \\
&\qquad a_S^j + b_S^j = 1\,,\ \forall\, S \subset N,\ \forall\, j \in M \\
&\qquad x(N) = E \\
&\qquad x_i \ge 0\,,\ i = 1, 2, \ldots, n \\
&\qquad a_S^j,\ b_S^j \in \{0, 1\},\ \forall\, S \subset N,\ \forall\, j \in M
\end{aligned}
$$

If the optimal solution is unique, then the procedure ends and this allocation is the MT rule. Otherwise, the optimal value of the problem (P.1) is denoted as $\boldsymbol{\epsilon_1}$, and the set of constraints that are active for any optimal solution as τ_1.

Step 2: Replace the active constraints obtained in Step 1 by equalities, and solve the following problem (P.2):

$$
\begin{aligned}
&\min \epsilon \\
&s.t: (E - c^j(N \backslash S))a_S^j - x(S) = \epsilon_1,\ \forall\, S \in \tau_1,\ \forall\, j \in M \\
&\qquad (E - c^j(N \backslash S))a_S^j - x(S) \le \epsilon,\ \forall\, S \notin \tau_1\,,\ S \subset N,\ \forall\, j \in M \\
&\qquad (E - c^j(N \backslash S)) - a_S^j K < 0,\ \forall\, S \subset N,\ \forall\, j \in M \\
&\qquad (E - c^j(N \backslash S)) + b_S^j K \ge 0,\ \forall\, S \subset N,\ \forall\, j \in M \\
&\qquad a_S^j + b_S^j = 1\,,\ \forall\, S \subset N,\ \forall\, j \in M \\
&\qquad x(N) = E \\
&\qquad x_i \ge 0\,,\ i = 1, 2, \ldots, n \\
&\qquad a_S^j,\ b_S^j \in \{0, 1\},\ \forall\, S \subset N,\ \forall\, j \in M
\end{aligned}
$$

If the optimal solution of the problem (P.2) is unique, then the procedure ends. Otherwise, the optimal value of the problem (P.2) is denoted by $\boldsymbol{\epsilon_2}$, and the set of constraints that are active for any optimal solution by τ_2.

. . . . . .

If Step $n - 1$ is reached and the algorithm has failed to stop, then:

Step n: Denote by ϵ_{n-1} the optimal value of the problem in Step $n - 1$. By changing active constraints of the problem obtained in Step $n - 1$ into equalities with $\epsilon = \epsilon_{n-1}$, then the following linear program (P.n) is obtained and solved:

$$
\begin{array}{ll}
\min \epsilon & \\
s.t: & (E - c^j(N\backslash\{S\}))a_S^j - x(S) = \epsilon_1, \forall S \in \tau_1, \ \forall j \in M \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& (E - c^j(N\backslash\{S\}))a_S^j - x(S) = \epsilon_{n-1}, \forall S \in \tau_{n-1}, \ \forall j \in M \\
& (E - c^j(N\backslash\{S\}))a_S^j - x(S) \leq \epsilon, \forall S \notin \tau_1 \cup \tau_2 \cup \cdots \cup \tau_{n-1}, \forall S \subset N, \ \forall j \in M \\
& (E - c^j(N\backslash\{S\})) - a_S^j K < 0, \forall S \subset N, \ \forall j \in M \\
& (E - c^j(N\backslash\{S\})) + b_S^j K \geq 0, \forall S \subset N, \ \forall j \in M \\
& a_S^j + b_S^j = 1\,, \ \forall S \subset N, \ \forall j \in M \\
& x(N) = E \\
& x_i \geq 0\,, \ i = 1, 2, \ldots, n \\
& a_S^j, \ b_S^j \in \{0, 1\}, \ \forall S \subset N, \ \forall j \in M
\end{array}
$$

If the optimal solution of problem (P.n) is unique, then the procedure ends, otherwise, the algorithm continues analogously, saturating the active constraints in each level until a problem with a unique solution is obtained.

In each step there is at least one active constraint which is replaced by equality. After n steps there are at least $n-1$ equality constraints of this kind, which, along with the efficiency constraint, $x(N) = E$, uniquely determine one unique payoff vector $\mathbf{x}^*$. Therefore the algorithm for at most n steps.

Remark 1 *The justification for the direction of inequalities in certain constraints of the linear program, defined in the proposed algorithm, is as follows:*

- *When the amount* $\left(E - c^j(N \setminus S)\right), \forall S \subset N, \ j \in M$, *is greater than or equal to zero, then the dissatisfaction for coalition* S *with respect to* $\boldsymbol{x}$ *is* $\left(E - c^j(N \setminus S)\right) - x(S)$, *the variable* a_S^j, *has to value 1 (and therefore* b_S^j *has to take value 0). For this to happen, the corresponding constraint is of strict inequality (see for example, constraint number 2 in problem (P.1)).*
- *When the amount* $\left(E - c^j(N \setminus S)\right), \forall S \subset N, \ j \in M$, *is strictly negative, then the dissatisfaction for coalition* S *with respect to* $\boldsymbol{x}$ *is* $-x(S)$, *and the variable* a_S^j *has to equal 0, which is equivalent to* b_S^j *taking value 1. This is obtained by making the corresponding constraint greater than or equal (see constraint number 3 in (P.1)).*

The iterative procedure proposed here to obtain the nucleolus, presents certain difficulties. It requires (in each iteration), identification of the set of constraints which are active for any optimal solution of the problem, and furthermore, the number of constraints in each problem is exponential in n.

Our goal is to reflect the complete path of awards of the MT rule. To this end, the aforementioned iterative procedure is applied, with the advantage that it will only be necessary to compute the nucleolus for a reduced number of amounts to divide. As shown in Hinojosa et al. (2012), the path of awards of the MT rule is continuous and piecewise linear. This advantage enables our propose procedure to obtain such a path of awards by simply computing the nucleolus for certain amounts, those in which the path of awards has a slope change. The remaining allocations of rule MT can be easily deduced by means of an algebraic expression.

In order to construct the procedure which enables the allocations given by MT rule in any division problem with multiple references, $(C, E) \in \mathcal{C}_N^M$, the analysis is used of the path of awards of the MT rule indicated in Section 3.2. According to this analysis, when the amount to divide does not exceed $\underline{c}(N)$, i.e. $E \leq \underline{c}(N)$, then the MT rule provides the same vector allocations as the Talmud rule for a classic division problem with references equal to $\underline{\mathbf{c}}$. However, when $E > \underline{c}(N)$, the behaviour of the rule changes with respect to the scalar case, since the surplus is not divided equally among the agents. A procedure for computing such allocations is proposed here.

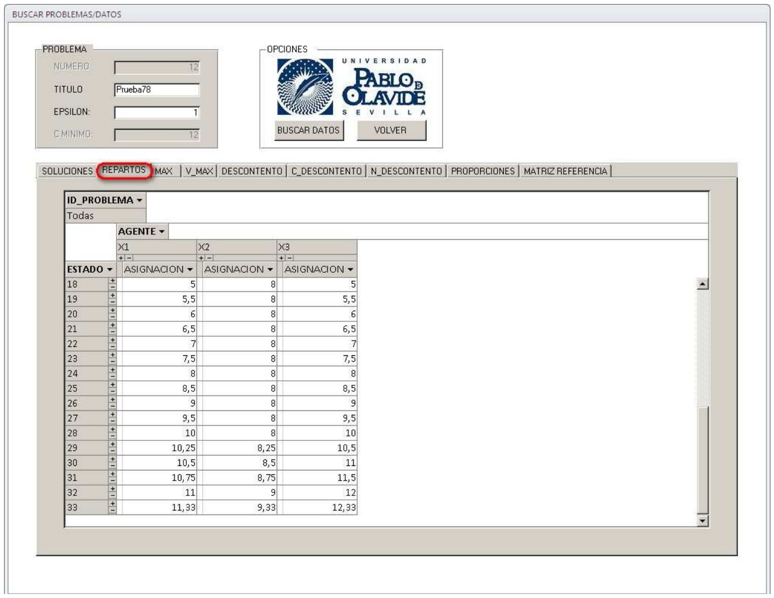
The algorithm starts by computing the allocation of the MT rule to divide an amount equal to $\underline{c}(N)$. In this case $\mathbf{x}^0 = MT(C, E) = \underline{\mathbf{c}}$. The procedure obtains a surplus of coalitions, $e(\mathbf{x}^0, S) = v^{\max}(S) - x^0(S)$ with respect to this allocation, $\mathbf{x}^0$. This surplus, for each coalition $S \subset N$, enables its level of dissatisfaction, $L^{\mathbf{x}^0}(S)$, at $\mathbf{x}^0$ to be obtained. For example, $L^{\mathbf{x}^0}(S) = 1$ indicates that coalition $S \subset N$ is the most dissatisfied (it has the largest $e(\mathbf{x}^0, S)$) at $\mathbf{x}^0$. We can say that such a coalition is in the first level of dissatisfaction. $L^{\mathbf{x}^0}(S) = 2$ means that coalition $S \subset N$, is the second most dissatisfied with respect to $\mathbf{x}^0$, i.e. this coalition is in the second level of dissatisfaction, and so on. Note that there can be more than one coalition with the same level of dissatisfaction. Coalitions which have the same level of dissatisfaction with $\mathbf{x}^0$ are denoted as $\mathcal{S}_l^{\mathbf{x}^0}$. All coalitions in $\mathcal{S}_1^{\mathbf{x}^0}$ have the same level of dissatisfaction and represent those groups with the maximum surplus at $\mathbf{x}^0$. All the coalitions in $\mathcal{S}_2^{\mathbf{x}^0}$ have the second maximum surplus at $\mathbf{x}^0$, and so on. Individual coalitions of agents dissatisfied with $\mathbf{x}^0$, $F^{\mathbf{x}^0} = \{i \in N \,|\, e(\mathbf{x}^0, \{i\}) \geq e(\mathbf{x}^0, \{j\}) \, \forall \, j \in N\}$, are also computed. When the amount to divide is larger than $E^* = \max \left\{\underline{c}(N), \max_{i\in N} \min_{j\in M}\{c^j(N \setminus \{i\})\}\right\}$ and $F^{\mathbf{x}^0} = N$, then the algorithm ends and each increment of the amount to be distributed is divided equally among the agents (see Hinojosa et al. (2012)). Otherwise, if neither of the two conditions above is satisfied, then the algorithm computes the nucleolus (by applying the iterative method previously proposed) for a new amount, $E^{\mathbf{x}} = E^{\mathbf{x}^0} + \varepsilon$. This allocation is denoted by $\mathbf{x} = MT(C, E)$, and is applied to calculate increasing the proportion of the amount to be divided, which is then assigned to each agent according to the MT rule. This is denoted as $\alpha^{\mathbf{x}^0}$, namely, $\alpha^{\mathbf{x}^0} = \frac{\mathbf{x}-\mathbf{x}^0}{\varepsilon}$. Note that this n-dimensional vector verifies $\sum_{i=1}^n \alpha_i^{\mathbf{x}^0} = 1$. $A^{\mathbf{x}^0}$ is the maximum increment of the amount to divide for which two coalitions with the same level of dissatisfaction in $\mathbf{x}^0$, maintain such a level. In the interval of the amount to divide which ranges from $E^{\mathbf{x}^0}$ to $E^{\mathbf{x}^1} = E^{\mathbf{x}^0} + A^{\mathbf{x}^0}$, the proportion $\alpha^{\mathbf{x}^0}$ remains constant. Therefore, any allocation of the MT rule in this interval can easily be deducted, as follows: For each $E^{\mathbf{x}} \geq E^{\mathbf{x}^0}$ and each $0 < \varepsilon < \varepsilon' < A^{\mathbf{x}^0}$ with $E^{\mathbf{x}'} = E^{\mathbf{x}} + \varepsilon$ and $E^{\mathbf{x}''} = E^{\mathbf{x}} + \varepsilon'$, if $\mathbf{x}' = \mathbf{x} + \varepsilon\alpha^{\mathbf{x}^0}$ then $\mathbf{x}'' = \mathbf{x} + \varepsilon'\alpha^{\mathbf{x}^0}$. In $E^{\mathbf{x}^1}$, the slope of the path of awards of the MT rule can change, and hence it is necessary to recompute the nucleolus for the amount $E^{\mathbf{x}} = E^{\mathbf{x}^1} + \varepsilon$. This allocation allows the computation to be performed in an analogous way to that described above: $\alpha^{\mathbf{x}^1}$, $A^{\mathbf{x}^1}$, $L^{\mathbf{x}^1}(S)$, $\mathcal{S}_l^{\mathbf{x}^1}$ and $F^{\mathbf{x}^1}$. Note that the set $\{\mathbf{x}^0 = \underline{\mathbf{c}}, \mathbf{x}^1, \mathbf{x}^2, \ldots\}$ includes the allocations in the path of awards of the MT rule where the slope of this path of awards can change. The procedure ends when $F^{\mathbf{x}^k} = N$ (for a certain k) and the amount to divide is greater than E^*.

An algorithm to compute the complete path of awards of the MT rule is described below:

Algorithm

```
IF E ≤ c(N)
        x = T(c, E)
ELSE
        k ← 0
        x^k ← c
        E^{x^k} ← c(N)
        F^{x^k} ← F^c
        β^{x^k} ← (1/n, 1/n, ..., 1/n)^t
        E* ← max { c(N), max_{i∈N} min_{j∈M} {c^j(N \ {i})} }
        WHILE E^{x^k} < E* or F^{x^k} ≠ N DO
                Compute α^{x^k}, A^{x^k}, y = x^k + A^{x^k} α^{x^k},
                E^y = E^{x^k} + A^{x^k}, F^y
                β^{x^k} ← α^{x^k}
                IF E ≤ E^y
                        x = x^k + (E − E^{x^k}) β^{x^k}
                ELSE
                        k ← k + 1
                        x^k ← y
                        E^{x^k} ← E^y
                        F^{x^k} ← F^y
                END IF
        END WHILE
        x = x^k + (E − E^{x^k}) β^{x^k}
END IF
```

We have designed a computer program which implements the procedure that describes the complete path of awards of the MT rule. This program is employed to obtain the complete path of awards of the MT rule in Example 1. The results are shown in Fig. 2.



ESTADO	X1 ASIGNACION	X2 ASIGNACION	X3 ASIGNACION
18	5	8	5
19	5,5	8	5,5
20	6	8	6
21	6,5	8	6,5
22	7	8	7
23	7,5	8	7,5
24	8	8	8
25	8,5	8	8,5
26	9	8	9
27	9,5	8	9,5
28	10	8	10
29	10,25	8,25	10,5
30	10,5	8,5	11
31	10,75	8,75	11,5
32	11	9	12
33	11,33	9,33	12,33

Fig. 2. Path of awards of the MT rule.

For three agents the path of awards of the MT rule can be represented graphically (see Fig. 3.).

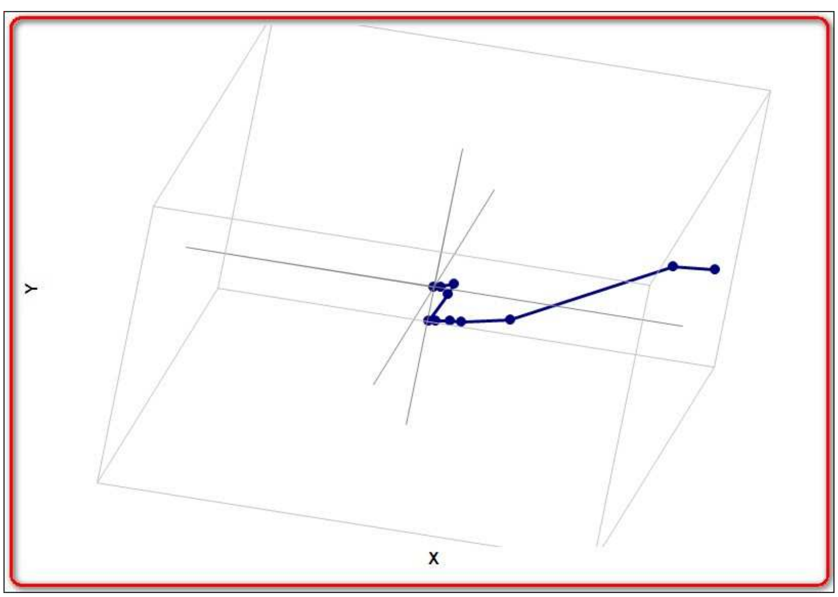


Fig. 3. Path of awards of the MT rule.

5. Conclusions

We have addressed the extension of the classic division problem with claims to situations in which the agents involved are characterized by several parameters instead of by a single parameter. For this class of problems the nucleolus is analyzed, and an algorithm to compute the allocation provided by the MT rule in a division problem with multiple references is described. This algorithm has been used to obtain the path of awards of the MT rule. This rule has a continuous, piecewise linear path of awards, and this allows the procedure presented to calculate the nucleolus only in certain quantities: Those in which the path of awards of the rule has a slope change. The remaining path of awards can easily be deduced through a simple algebraic expression.

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References

1. M. A. Hinojosa, A. M. Mármol, F. Sánchez, "A Consistent Talmudic Rule for Division Problems with Multiple References". *TOP (Revista de Inv. Operativa de la Soc. Española de Estadística e Inv. Operativa)* (2012), Vol. 20, Number 3, 661–678. DOI: 10.1007/s11750-010-0158-4.
2. R. Aumann, M. Maschler, "Game Theoretic Analysis of Bankruptcy Problem from the Talmud", *Journal of Economic Theory*, (1985), Vol. 36, 195–213.
3. X. Deng, CH. Papadimitriou, " On the complexity of cooperative game solution concepts", *Mathematics of Operations Research*, (1994), Vol. 19, 257–266.
4. X. Deng, T. Ibaraki, H. Nagamochi, "Combinatorial optimization games", *Proc. 8th Annual ACMSIAM Symposium on Discrete Algorithms*, New Orleans, LA, (1997), 720–729.
5. D. Granot, F. Granot, W.R. Zhu, "Characterization sets for the nucleolus", *International Journal of Game Theory*, (1998), Vol. 27, 359–374.
6. M. Maschler, B. Peleg, L.S. Shapley, "Geometric Properties of the Kernel, Nucleolus, and Related Solution Concepts ", *Mathematics of Operations Research*, (1979), Vol. 4, n 4, 303–338.
7. G. Owen, "Game Theory", *Academic Press*, San Diego, California (1995).
8. B. O'Neill, "Game Theoretic Analysis of Bankruptcy Problem from the Talmud", *Mathematics Social Sciences*, (1982), Vol. 2, 345–371.
9. P. Young, "Distributive Justice in Taxation", *Journal of Economic Theory*, (1988),Vol. 44, 321–335.
10. P. Young, "Progressive Taxation and Equal Sacrifice", *American Economic Review*, (1990), Vol. 80, 253–266.
11. H. Moulin "Equal or Proportional Division of a Surplus, and Other Methods", *International Journal of Game Theory*, (1987), Vol. 16, 161–186.
12. W. Thomson, "Axiomatic and Game-Theoretic Analysis of Bankruptcy and Taxation Problems: a Survey", *Mathematical Social Sciences*, (2003), Vol. 45, 249–297.
13. D. Schmeidler, "The Nucleolus of a Characteristic Function Game", *SIAM Journal of Applied Mathematics*, (1969), Vol. 16, 1163–1170.
14. M. Maschler, "The Bargaining Set, Kernel and Nucleolus", *In Handbook of Game Theory with Economics Applications (R.j. Aumann y S. Hart, Eds.)*, (1992), Vol. I. Amsterdam: Elservier Science Publishers.
15. R. Serrano, "Strategic Bargaining, Surplus Sharing Problems and the Nucleolus", *Journal of Mathematical Economics*, (1995), Vol. 24, 319–329.
16. T. Solymosi, Tes. Raghavan, "An algorithm for finding the nucleolus of assignment games", *International Journal of Game Theory*, (1994), Vol. 23, 119–143.
17. N. Megiddo, "Computational complexity of the game theory approach to cost allocation for a tree", *Mathematics of Operations Research*, (1978), Vol. 3, 189-196.
18. J. Kuipers, "A polynomial time algorithm for computing the nucleolus of convex games", Report M 96-12, Maastricht University, Maastricht, the Netherlands, (1996).
19. U. Faigle, S. Fekete, W. Hochstttler, W. Kern, "On the complexity of testing membership in the core of min cost spanning tree games", *International Journal of Game Theory*, (1997), Vol. 26, 361–366.

20. U. Faigle, W. Kern, J. Kuipers, "Computing the nucleolus in min cost spanning tree games is NP-hard", *International Journal of Game Theory*, (1998), Vol. 27, 443–450.
21. J.A.M. Potters, J.H. Reijnierse, M. Ansing, "Computing the nucleolus by solving a prolonged simplex algorithm", *Mathematics of Operations Research*, (1996), Vol. 21, 757-768.
22. E. Kohlberg, "The nucleolus as a solution of a minimization problem", *SIAM J. Appl. Math.*, (1972), Vol. 23, 34–39.
23. G. Owen, "A note on the nucleolus", *International Journal of Game Theory*, (1974), Vol. 3, 101–103.
24. J.K. Sankaran, "On finding the nucleolus of an n-person cooperative game", *International Journal of Game Theory*, (1991), Vol. 19, 329–338.
25. M. Leng, M. Parlar, "Analytic Solution for the Nucleolus of a Three-Player Cooperative Game", *Naval Research Logistics*, (2010), Vol. 57, n 7, 667–672.
26. J. Puerto, "A note on finding the nucleolus of an n-person cooperative game by a single linear program", Mimeo, University of Sevilla, (2011).