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RECTANGULAR INPUT-OUTPUT MODELS BY MOORE-PENROSE INVERSE

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ABSTRACT: The official available data must be the primary datasource in economic studies. However, in economic modelling related with supply-use tables this aspect is ignored. In fact, as in this structure the number of products exceeds the number of industries, there is considerable information lost due to aggregations in order to get square matrices. Nevertheless, rectangular matrices can directly be modelled using generalized inverses. The main aim of this paper is to highlight the fact that it is not required to elaborate symmetric input-output tables for modelling. In spite of it, in this paper it is reminded how to construct symmetric matrices through the two main traditional hypotheses: product technology and industry technology. Finally, an empirical application is shown with the goal of highlight different alternatives in this area.

Keywords: Moore-Penrose inverse, input-output, rectangular model, supply-use.

1. Introduction

The input-output framework (IO) offers numerous analytical possibilities because it is possible to work directly through supply-use tables (SUTs) or through the traditional symmetric table (ST). In this paper, we are going to focus on the general case. In fact, several models of demand and supply are developed within the SUTs structure.

Generally, intermediate consumption matrices related with SUTs framework are rectangular. Therefore, to obtain the production vector (by products or by industries), in the construction of models from the SUTs, it is necessary to work with the "inverses" of rectangular matrices. In order to avoid an assummed problem, there is a simple and quite employed procedure, which consists on aggregating products to achieve a square matrix. But undoubtedly, this implies a significant loss of information available on the SUTs. We want to highlight that, in economic research, IO inverses must be considered as a working tool and not like a final product. In this paper, we are going to show how this problem can be avoided using generalized inverses, providing the advantage that they are perfectly calibrated and, consequently, they can be used as an analytical tool. Recently, a previous study developed a demand model whitin the SUTs framework through the use of the Moore-Penrose inverse [1]. The present paper aims to go deeper into the rectangular IO models in order to highlight their true potential.

One of the most important characteristics in the economic analysis is that available official data must be used as the primary datasource, in the best way possible. Therefore, losing information by aggregations is not desirable. We choose to calculate rectangular inverses matrices through the generalized Moore-Penrose method, [2] [3]. There are other generalized inverses, like the one of Drazin, but in this paper we select the Moore-Penrose one due to its flexibility. This tool is widely used in other fields, although in the economics context, it was also used in various subjects. For example, it was used in dynamic demand models [5], in econometrics regarding partitioned matrices [4], in aggregating industries [6] or in social accounting matrices [7]. It was also used in the SUTs framework to avoid aggregations in demand models based on the assumption of product technology [8].

Regarding the use of all the available information, it would be interesting to explain the possibility of modeling using also the global updating of SUTs [9]. This combination of techniques would facilitate better economic estimations through rectangular schemes.

2. Preliminary issues

There are different possibilities of IO analysis depending on the character of the accounting representations: rectangular or symmetrical. At the same time, there are models expressed in production by product or industry. In fact, what is done is build ST, in one case product by product and the other, sector by sector. That is to say, there are various analysis options using SUTs. To this effect, see the manuals of the UN and Eurostat, [10] [11]. On this occasion, the models of domestic flows will be addressed given the interest that they have in regional studies, although the mathematical developments are analogous for the elaboration of total flows.

It is understood that remembering the definition of the pseudoinverse concept by Moore-Penrose would be correct. To this effect, the matrix $A \in M_{m\times n}(\Re)$ is considered, it is said that $A^{\dagger}$ (matrix $n\times m$) is the Moore-Penrose pseudoinverse, if, and only if the following properties are verified:

$$AA^{\dagger}A = A \text{ (P.1)}, AA^{\dagger} \text{ is symmetric (P.2)}, A^{\dagger}A \text{ is symmetric (P.3) and } A^{\dagger}AA^{\dagger} = A^{\dagger} \text{ (P.4)};$$

likewise $A^{\dagger}$ always exists and is unique. It also holds that

- If $m \geq n$ and $rg(A) = n$ then $A^{\dagger} = (A^{T}A)^{-1}A^{T}$ and $A^{\dagger}A = I_n$.
- If $m \leq n$ and $rg(\mathrm{A}) = m$ then $A^{\dagger} = A^{T}(AA^{T})^{-1}$ and $AA^{\dagger} = I_m$.
- If $rg(A) = r \leq \min\{m, n\}$ then $A^{\dagger}=C^{\dagger}B^{\dagger}$, $B^{\dagger}$ and $C^{\dagger}$ being the pseudoinverse of the matrices $B \in M_{m\times r}(\Re)$ and $C \in M_{r\times n}(\Re)$ of range r such that $A = BC$.

Hereinafter, i corresponds to a column vector with all of its components equal to one. The symbol ^{T} denotes the transformation and $^{-1}$ the inversion. As these operations switch with one other, their composition is expressed as $^{-T}$. The notation ^ reflects the diagonalization of the associated vector, and it is understood that the elements that do not belong to the main diagonal are equal to zero. Additionally, in the matrix equations the uppercase notations refer to matrices and the lowercase to vectors.

Two accounting relationships from the production matrix V can be defined. On one hand, production by product is obtained as the sum of rows

$$q = Vi. \tag{1}$$

And on the other hand, the production by industry corresponds to the sum of columns of said matrix,

$$g = V^T i. \tag{2}$$

From the definitions of the technical coefficients and distribution (internal in both cases), it is immediately seen how the matrices that encompass said coefficients B^d and H^d, respectively, are expressed in the following way:

$$B^d = U^d \hat{g}^{-1} \tag{3}$$

$$\text{and} \quad H^d = \hat{q}^{-1} U^d. \tag{4}$$

In turn, the market coefficients matrix results from the following matrix product:

$$D = \hat{q}^{-1} V, \tag{5}$$

after certain replacements, we obtain that

$$g = D^T q. \tag{6}$$

The specialization coefficients matrix is built as indicated below:

$$C = V \hat{g}^{-1}, \tag{7}$$

and from there, it is known that

$$q = Cg. \tag{8}$$

3. Supply-use models: different alternatives

To develop models for products using SUTs, it is necessary to use simplifying assumptions that establish two types of relationships between production by product and by sector. In one case it is necessary to rely on the stability of the the production matrix structure by row and, in the other case the stability of the structures of the same matrix by column is considered. In summary, there are two courses of action: models based on assumed industry technology and based on assumed product technology.

In relation to the industry technology model, the market coefficient matrix, D, is stable temporarily. That is, the assumption of constant market share of a product for different branches are always used. Therefore, the equality for products is:

$$q = U^d i + y^d, \tag{9}$$

where y^d is the final domestic demand for products and U^d is the matrix of internal intermediate consumption.

Continuing on, it is possible to substitute domestic intermediate demand $U^d i$, for $B^d g$, according to the stability of technical coefficients. That is to say

$$q = B^d g + y^d, \tag{10}$$

where B^d is the matrix of non-homogeneous internal technical coefficients.

However, based another the other working hypothesis, g can be substituted for $D^T q$. Consequently,

$$q = B^d D^T q + y^d. \tag{11}$$

Thus, it is possible to develop the demand model corresponding to the production by product:

$$q = (I - B^d D^T)^{-1} y^d. \quad (12)$$

In the product technology assumption models, the stability of the specialization coefficients matrix C is considered. Therefore, each product is produced with a specific technology, independent of the industry in which it is produced; in other words, the structure of intermediate consumption is that of the product. When the intermediate consumption and production matrices are square and of complete range alternative models can also be designed[1]. If so, the inverse of C is calculated; that is to say, the production by industry is given as $g = C^{-1}q$. So that alternately another demand by product model is obtained:

$$q = (I - B^d C^{-1})^{-1} y^d. \quad (13)$$

From an axiomatic point of view, the product technology assumption is theoretically superior to the other hypothesis [12]. In this sense, it is always possible to find a non-negative technical coefficient matrix that is consistent with the information available in the SUTs, [13] [14]. Furthermore, when using information at an establishment level regarding consumption and production, it has been empirically shown that the product technology assumption can also be useful [15]. However, the application of said hypothesis has its limitations [16]. On one hand, negative technical coefficients easily appear and secondly, it is absolutely necessary that the number of products be exactly the same as the number of industries, although this last aspect can be taken on using the pseudoinverses as will be shown below. Recently, the use of econometric models have been proposed for performing this type of analysis [17]. In any case, it should be noted that all of these problems have been treated with great intensity, [13] [14] [18] [19] [20] [21] [22] [23].

It is necessary to previously resolve whether the technology is that of the product or the industry, given that many products are made by several industries and many industries produce more than one product. There are theoretical difficulties in sustaining the technology assumption in an industry being that it means the same production cost for different products sold at different prices. In practice, this hypothesis is the most widely used due to the alleged inability to invert the matrix of specialization coefficients. Over time, the use of models based on this technology is also problematic because it implies the stability of market shares. The specialization coefficients admittedly vary to a lesser extent.

In any case, it is not always necessary to build STs in the vicinity of SUTs, as it is feasible to get a rectangular model through the product technology assumption, without the obligation to use aggregations. It is easily explained because q can be substituted for Cg in the previous system, then

$$Cg = B^d g + y^d, \quad (14)$$

or, alternatively,

$$(C - B^d)\, g = y^d. \quad (15)$$

To isolate g, the Moore-Penrose pseudoinverse is used. In general, matrix $(C - B^d)$ is rectangular and can be assumed that its range coincides with the number of columns. Additionally, in relation to the model that is intended to be constructed, matrix $(C - B^d)$ needs to have a greated number of rows than columns, that way the pseudoinverse $(C - B^d)^{\dagger}$, is of the order $n \times m$ and $(C - B^d)^{\dagger} (C - B^d) = I_n$[2]. Then, by simplifying, the internal flow model is obtained relative to the production by industries[3]:

$$g = (C - B^d)^{\dagger} y^d. \quad (16)$$

4. Simplified supply model

Supply models have been presented as a natural variant of demand models, this alternative being proposed by Ghosh, [24]. The plausability of this supply model is not being addressed here, this characteristic has been the object of debate in many situations, [25] [26] [27] [28] [29] [30] [31]. In this section the supply model is presented only in the context of SUTs.

[1] In general, these matrices are not square, except when aggregations are used by row. According to the construction of the SUTs, there are more products than industries.

[2] If $m=n$ this problem disappears, in the particular case of inversed matrices.

[3] For more details see [1].

The following accounting relationship is fundamental for the construction of these models:

$$(U^d)^T i + (U^m)^T i + v = g, \qquad (17)$$

where U^m is the imported intermediate consumption and v the vector of primary input. In order to simplify the formulas $t^m = (U^m)^T i$ is considered, which is the intermediate input vector of imported flows.

From here, taking into account that $(U^d)^T i = (H^d)^T q$, which implies the stability of (non-homogenous) internal distribution coefficients, we arrive at

$$(H^d)^T q + t^m + v = g. \qquad (18)$$

Then, assuming the stability of matrix, D, g can be substituted for $D^T q$

$$(H^d)^T q + t^m + v = D^T q, \qquad (19)$$

then, it is operated in order to solve production by product, q. Therefore, the system should be expressed as follows:

$$[D^T - (H^d)^T]\, q = t^m + v. \qquad (20)$$

Now, the Moore-Penrose pseudoinverse is used, in which the members of the previous equality are pre-multiplied by $[D^T - (H^d)^T]^\dagger$. So that

$$[D^T - (H^d)^T]^\dagger [D^T - (H^d)^T]\, q = [D^T - (H^d)^T]^\dagger (t^m + v), \qquad (21)$$

although it is necessary that the SUTs possess a greater number of columns than rows, or in some cases cases an equal number, ensuring that

$$[D^T - (H^d)^T]^\dagger [D^T - (H^d)^T] = I_m, \qquad (22)$$

if so, the supply model corresponding to production by products is obtained as:

$$q = [D^T - (H^d)^T]^\dagger (t^m + v), \qquad (23)$$

and the associated model would be the following:

$$g = D^T [D^T - (H^d)^T]^\dagger (t^m + v). \qquad (24)$$

Similarly, the simplified supply model for total flow products is introduced. As will be seen later, the independent variable is v in this context. So the equality that is considered is

$$U^T i + v = g \qquad (25)$$

and considering the hypothetical stability of D and H, g can be substituted for $D^T q$ and the vector of intermediate input for $H^T q$. In such a way that we obtain

$$H^T q + v = D^T q. \qquad (26)$$

Now, in order to clear the production by product, we operate

$$(D^T - H^T)\, q = v, \qquad (27)$$

then both members of the previous equality are pre-multiplied by the pseudoinverse of $(D^T - H^T)$ –full range matrix and of the order $m \times n$, with the particularity that $m < n$– and the equation of the model is immediately obtained as

$$q = (D^T - H^T)^\dagger v. \qquad (28)$$

In relation to the supply models, it is also feasible to use the transformation. In this sense, the equality between input and production (in its transformation) is considered,

$$iU + v^T = g^T, \qquad (29)$$

and thereafter the above substitutions are used (although adapted to this scenario), it is seen how the initial equality can be alternately displayed as:

$$q^T H + v^T = q^T D. \qquad (30)$$

Then, with the intention of explaining production by product, it is operated in such a way as to obtain

$$q^T(D - H) = v^T, \quad (31)$$

from here, we must post-multipy both members of the equality by the pseudoinverse of $(D - H)$. This matrix must be $m \times n$, which is to say the number of products is strictly less than or equal to the number of industries, hence

$$(D - H)(D - H)^{\dagger} = I_m. \quad (32)$$

Therefore, the resulting supply model is:

$$q^T = v^T(D - H)^{\dagger}. \quad (33)$$

The matrix $(D - H)^{\dagger}$ is Ghosh´s pseudoinverse and its generic element, δ_{ij}, represents the increase in the production of product j of one unit of the added value of industry i.

5. Transformation of supply-use tables in symmetric tables

In this section we deal with the different estimates of square matrices of internal inputs[4] from the internal consumption matrix U^d, from the two possible perspectives. Additionally, the same estimate is obtained, hence there is a complementarity between demand and supply.

If, for any reason, the intention is to estimate technical coefficient matrices (or distribution) associated with the models built from the SUTs computing would also be immediate. Diffrent studies explain the way these technical coefficient matrices are obtained from the perspective of demand, [16]. But when using the technology assumption of a product they work with square schemes in which information available in the SUTs is lost, as noted. For the aforementioned reasons, when the product technology assumption is used supply and demand models cannot be constructed at the same time if the number of products and industries do not coincide, from then on it is characterized by having the same fixed number of rows as columns, $m = n$.

To get a product by product square matrix of internal intermediate consumption based on the industry technology assumption, $W^d(i)$, there are two alternatives that lead to the same solution, via demand or via supply. Reference models are (12) and

$$q = [I - C(H^d)^T]^{-1}C(t^m + v). \quad (34)$$

Then, it is easily proven that the following equality is carried out

$$B^dD^T\hat{q} = [C(H^d)^T\hat{q}]^T. \quad (35)$$

On one hand, it must be

$$B^dD^T\hat{q} = U^d\hat{g}^{-1}V^T\hat{q}^{-1}\hat{q} \quad (36)$$

and simplifying would give us

$$W^d(i) = U^d\hat{g}^{-1}V^T = U^dC^T. \quad (37)$$

On the other hand, considering the property relative to the transformation of the product of matrices it must be

$$[C(H^d)^T\hat{q}]^T = \hat{q}H^dC^T \quad (38)$$

and it looks like

$$\hat{q}H^dC^T = \hat{q}\hat{q}^{-1}U^d(V\hat{g}^{-1})^T. \quad (39)$$

Finally, when operating $W^d(i)$ is obtained from this point

$$W^d(i) = U^d\hat{g}^{-1}V^T = U^dC^T. \quad (40)$$

[4] Hereinafter, the product by product (internal) intermediate consumption matrix is symbolized as W^d, and if it is sector by sector as ω^d. The working hypothesis is also specified in parenthesis: technology of product r industry, (*o*) or (*i*) as appropriate.

Regarding the estimation of a square matrix of internal intermediate input by industry according to the aforementioned assumptions, $\omega^{d}(i)$, something analogous to what is shown above occus. It can also be approached from the two perspectives indicated. The reference models are the following:

$$g = (I - D^{T} B^{d})^{-1} D^{T} y^{d} \tag{41}$$

and $$g = [I - (H^{d})^{T} C]^{-1}(t^{m} + v). \tag{42}$$

In effect, as will be seen in the following equality relative to internal intermediate consumption:

$$\omega^{d}(i) = D^{T} B^{d}\, \hat{g} = [(H^{d})^{T} C\, \hat{g}\,]^{T}. \tag{43}$$

In order to check it, the first member is modified depending on the corresponding coefficients to the SUTs. Specifically,

$$D^{T} B^{d}\, \hat{g} = V^{T}\, \hat{q}^{\,-1} U^{d}\, \hat{g}^{\,-1}\, \hat{g} = V^{T}\, \hat{q}^{\,-1} U^{d}, \tag{44}$$

but as $V^{T}\, \hat{q}^{\,-1}$ corresponds with D^{T}, and via demand the following is obtained

$$\omega^{d}(i) = D^{T} U^{d}. \tag{45}$$

At the same time, it must be

$$\hat{g}\, C^{T} H^{d} = \hat{g}\,(V\, \hat{g}^{\,-1})^{T}\, \hat{q}^{\,-1} U^{d} \tag{46}$$

and operating, we are left with

$$\hat{g}\;(V\, \hat{g}^{\,-1})^{T}\; \hat{q}^{\,-1}\; U^{d} = \hat{g}\; \hat{g}^{\,-1} V^{T}\; \hat{q}^{\,-1} U^{d}. \tag{47}$$

Now, simplifying

$$\hat{g}\; \hat{g}^{\,-1} V^{T}\; \hat{q}^{\,-1} U^{d} = V^{T}\; \hat{q}^{\,-1} U^{d}, \tag{48}$$

that is, as expected, from the perspective of supply the same expression is reached. Consequently,

$$V^{T}\; \hat{q}^{\,-1} U^{d} = D^{T} U^{d}. \tag{49}$$

In reality, $\omega^{d}(i)$ corresponds with the product of the transformation of the market coefficients for the internal intermediate consumption matrix.

According to the product technology assumption and in the context of the same number of products and sectors, we also see how the corresponding supply and demand models are complementary, and respectively are (13) and

$$q = [I - D^{-T}(H^{d})^{T}]^{-1} D^{-T}(t^{m} + v). \tag{50}$$

It is observed, on one hand, how the product by product intermediate consumption matrix would be, $W^{d}(o)$, in relation to the demand model:

$$W^{d}(o) = B^{d} C^{-1}\, \hat{q} = U^{d}\; \hat{g}^{\,-1}(V\; \hat{g}^{\,-1})^{-1}\; \hat{q}\,, \tag{51}$$

Considering the property of the inverse of a product must be

$$(V\; \hat{g}^{\,-1})^{-1} = \hat{g}\, V^{-1}, \tag{52}$$

substituting as

$$U^{d}\; \hat{g}^{\,-1}(V\; \hat{g}^{\,-1})^{-1}\; \hat{q} = U^{d}\; \hat{g}^{\,-1}\; \hat{g}\, V^{-1}\; \hat{q} \tag{53}$$

and, now simplifying we get

$$U^{d}\; \hat{g}^{\,-1}\; \hat{g}\, V^{-1}\; \hat{q} = U^{d}(\hat{q}^{\,-1} V)^{-1} = U^{d} D^{-1}. \tag{54}$$

Therefore,

$$W^{d}(o) = B^{d} C^{-1}\, \hat{q} = U^{d} D^{-1}. \tag{55}$$

In relation to this model, the associated internal technical coefficients, A_o (U^d, V), are immediately obtained. In effect,

$$A_o(U^d, V) = B^dC^{-1} = U^d\hat{g}^{-1}(V\hat{g}^{-1})^{-1} = U^d\hat{g}^{-1}\hat{g}V^{-1} = U^dV^{-1}, \quad (56)$$

this result appears in various publications, although for total flows, [32] [22][16].

On the other hand, we see that the intermediate consumption matrix relative to the supply model is

$$[D^{-T}(H^d)^T\hat{q}]^T = [(D^{-1})^T(H^d)^T\hat{q}]^T = \hat{q}H^dD^{-1} = \hat{q}\hat{q}^{-1}U^dD^{-1} = U^dD^{-1}. \quad (57)$$

From there, it can be expressed as follows:

$$W^d(o) = U^dD^{-1}. \quad (58)$$

The remaining possibilities are the supply and demand models relative to the production of sectors based on the product technology assumption. The symmetric scenario $m = n$ is chosen. The expression of these models is the following:

$$g = (I - C^{-1}B^d)^{-1}C^{-1}y^d \quad (59)$$

$$\text{and} \quad g = [I - (H^d)^TD^{-T}]^{-1}(t^m + v). \quad (60)$$

This shows their complementarity by estimating the intermediate consumption matrix sector by sector, $\omega^d(o)$, according to this assumption. If we now turn to the demand model indicated, the matrix comes from the following matrix product:

$$C^{-1}B^d\hat{g}. \quad (61)$$

From here, B^d is replaced and through a mere simplification we get

$$C^{-1}B^d\hat{g} = C^{-1}U^d\hat{g}^{-1}\hat{g} = C^{-1}U^d. \quad (62)$$

Through the supply model, the matrix $\omega^d(o)$ results from the transformation of the matrix product. Concretely,

$$[(H^d)^TD^{-T}\hat{g}]^T, \quad (63)$$

which can also be expressed alternatively through

$$\hat{g}D^{-1}H^d. \quad (64)$$

Taking into account that

$$D^{-1} = (\hat{q}^{-1}V)^{-1} = V^{-1}\hat{q} \quad (65)$$

and, at the same time, if H^d is replaced by $\hat{q}^{-1}U^d$ we get

$$\hat{g}D^{-1}H^d = \hat{g}V^{-1}\hat{q}\hat{q}^{-1}U^d = \hat{g}V^{-1}U^d, \quad (66)$$

but as $C = V\hat{g}^{-1}$ then it must be

$$C^{-1} = (V\hat{g}^{-1})^{-1} = \hat{g}V^{-1}, \quad (67)$$

in such a way that

$$\hat{g}V^{-1}U^d = C^{-1}U^d. \quad (68)$$

Definitively, the matrix in question is estimated through a matrix product:

$$\omega^d(o) = C^{-1}U^d. \quad (69)$$

In summary, the following table is presented in which the different alternatives for estimating internal intermediate consumption previously studied appear:

Table 1. Estimation of internal intermediate consumption

	Product technology assumption (m=n)	Industry technology assumption (m>n)
product × product	$W^d(o) = U^d D^{-1}$	$W^d(i) = U^d C^T$
industry × industry	$\omega^d(o)= C^{-1} U^d$	$\omega^d(i)= D^T U^d$

Source: own elaboration

Therefore, to obtain square (internal) intermediate consumption matrices for products the intermediate consumption matrix for internal flows, U^d, has to be post-multiplied by C^T or by D^{-1} (by one matrix or another depending on the assumed hypothesis). To obtain square intermediate consumption matrices for industries, U^d must be pre-multiplied by D^T or C^{-1} (once again, according to the considered hypothesis). Of course the correct multiplication for the pseudoinverse of D and C only make sense in certain contexts. Remember that the construction of models is determined by the number of products and industries, in one case $m>n$ is required and in the other $m<n$.

6. Practical application

In this section we turn to the SUTs of Euskadi from the year 2009 in order to highlight the usefulness of some methodological aspects addressed in this article. The Official Institute of Statistics of Euskadi (Eustat) offers the SUTs valued at basic prices with a disaggregation of 6×4. This dimension is very manageable and serves perfectly to highlight the main particularities of the rectangular format.

Thus, these tables are considered together and the (internal) Leontief pseudoinverse is calculated:

$$(C - B^d)^{\dagger} = \begin{bmatrix} 1{,}578 & 0{,}009 & 0{,}003 & -0{,}052 & 0{,}019 & 0{,}054 \\ -0{,}335 & 1{,}350 & 0{,}359 & 0{,}123 & 0{,}048 & 0{,}133 \\ 0{,}018 & 0{,}008 & 1{,}383 & 0{,}074 & 0{,}027 & 0{,}125 \\ 0{,}181 & 0{,}135 & 0{,}207 & 1{,}469 & 1{,}225 & 1{,}162 \end{bmatrix}$$

In general, this matrix has some specific characteristics. Unlike what happens in symmetric tables – where the elements of Leontief´s inverse are positive–, in this scenario negatives figures emerge. In this scheme, one product can be produced by different industries. By quantifying the alterations in the production, of a particular sector, motivated by modifications in final demand of a product, it is probable that certain substitutions emerge when developing the product. That is to say, that there can be a greater reliance on production in another sector, instead of the one considered. For this reason negative values probably appear. The presence of negative figures (relative to the product technology assumption[5]) has been studied in detail, and in general there are three reasons why these values are negative: measurement errors, coexistence of technologies, and/or aggregation problems, [16]. In any case, there are ways to correct these figures, such as the replacement of the negative values with zeros, then applying the RAS until the balance is reached, [32]. Another tool used to address this problem is the application of the Gauss-Jacobi algorithm, [33] [22]. The working hypothesis is product technology and the models founded on this hypothesis meet the four desirable axioms, but is very common that negative figures appear in the associated coefficient matrix, although they usually take very small values. The strong sectoral aggregation of an economy can also cause distortion in the results, which should be interpreted with caution.

5 The models can adopt different expressions but this problem is present in one case or another.

Following, we will check that the the simplified (internal) demand model is calibrated. In other words, that the matrix product of the previous pseudoinverse for final demand is equal to the production by (non homogeneous) industry.

1,58	0,01	0,00	–0,05	0,02	0,05
– 0,33	1,35	0,36	0,12	0,05	0,13
0,02	0,01	1,38	0,07	0,03	0,13
0,18	0,14	0,21	1,47	1,22	1,16

x

246.210
29.585.356
9.211.660
17.427.106
13.626.569
14.841.317

=

825.794
47.954.063
16.502.791
65.492.586

This is not to reproduce the different symmetrical matrices previously studied since their calculations are of no major problem. For example, intermediate consumption matrices are constructed through the industry technology assumption. In this sense, the internal intermediate consumption matrix is considered and the totals by row and column are highlighted:

U^d =

35.156	223.142	15.779	53.611	327.688
130.936	11.074.282	3.103.681	3.460.344	17.769.243
4.538	256.421	4.394.846	2.782.832	7.438.637
51.406	2.027.592	725.106	5.872.484	8.676.588
26.294	2.517.422	872.464	7.206.853	10.623.033
5.332	80.336	48.995	867.164	1.001.827
253.662	16.179.195	9.160.871	20.243.288	

First, the product by product matrix is indicated:

$W^d(i)$ =

23.710	229.262	16.559	23.516	21.619	13.023	327.688
89.243	10.859.499	3.128.578	1.483.392	1.369.892	838.638	17.769.243
3.778	265.142	4.372.882	1.096.083	1.027.895	672.856	7.438.637
36.179	2.022.748	734.736	2.312.570	2.150.170	1.420.184	8.676.588
19.603	2.498.847	884.467	2.838.345	2.638.905	1.742.866	10.623.033
3.818	84.311	49.983	338.813	315.229	209.672	1.001.827
176.332	15.959.809	9.187.205	8.092.720	7.523.711	4.897.240	

The the industry by industry matrix is shown:

	34.835	279.023	32.987	71.906	418.751
	130.976	11.029.784	3.141.782	3.705.519	18.008.061
$\omega^{d}(i)=$	4.608	261.903	4.333.326	2.768.426	7.368.263
	83.244	4.608.486	1.652.775	13.697.437	20.041.942
	253.662	16.179.195	9.160.871	20.243.288	

With this working hypothesis negative figures do not appear, nor is it necessary to resort to inverse (or pseudoinverse). The calculations are more basic than in the other hypthosis, here it is simply due to a sum of products with positive numbers.

7. Conclusions

In general, SUTs have a larger number of products than industries, i.e. they are presented in a rectangular structure. Although these formats can difficult economic modeling, it is not required to aggregate with the aim of obtaining square matrices. Rectangular matrices can be used by applying generalized inverse matrices, and therefore, alternative models of demand and supply, product by product or industry by industry, are obtained. This article has also highlighted that the rectangular models can be developed in a simple way, being calibrated and also applied.

During the paper, the construction of domestic symmetric matrices was explained following both the hypothesis of product technology and industry technology. Summarizing, to obtain the squared intermediate consumption matrices by products it is necessary to premultiply U^{d} by C^{T} or by D^{-1}, depending on the hypothesis. On the other side, in order to obtain the squared intermediate consumption matrices by industries it is necessary to postmultiply U^{d} by D^{T} or by C^{-1}. However, the construction of symmetric tables is not an essential procedure.

Briefly, we should not contemplate the elaboration of square matrices with the aim of just being inverted, but then to exploit all the information available in the SUTs. Moreover, it was highlighted throughout the paper that IO inverses must be considered as a working tool and not like a final product.

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