

NEW INDEXES FOR MEASURING ELECTORAL DISPROPORTIONALITY

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ABSTRACT: The number of representatives obtained by each political party in an electoral process must be a whole number. So, the percentage of votes for each party usually differs from the corresponding percentage of seats, forcing a certain unavoidable disproportionality. On the other hand, different elements of the electoral system (constituencies, thresholds, etc.) may produce some avoidable disproportionality. Those indexes traditionally used to analyse disproportionality take into account an unreachable exact proportionality as a reference. Instead, our more realistic approach quantifies distortions from a specific allotment, namely the seat distribution obtained when applying a proportional method to the total votes (that is, as if it were a unique constituency, without electoral thresholds or incentives to the winning party). Hence, we measure the avoidable disproportionality associated with such method. Unlike traditional indexes, we propose indexes associated with proportional allotment methods that can be zero in real situations. They are simple to calculate and allow us to decipher the number of seats assigned beyond the inevitable disproportionality which arises from the constraint of whole numbers. We are particularly interested in the indexes associated with Jefferson and Webster methods, which are compared to Gallagher, Loosemore-Hanby and Sainte-Laguë indexes for the results of 55 elections held in several countries.

Keywords: Proportional representation, disproportionality indexes, Loosemore-Hanby index, Jefferson method, Webster method.

1. Introduction

The electoral systems are mechanisms which translate votes into parliamentary seats. None of the procedures put into practice manages to faithfully reflect the general distribution of the voters' preferences. Even with the so called proportional systems, political parties do not receive seat percentages equivalent to their vote percentages. This fact generates what is known as an electoral disproportionality. Consequently some parties become overrepresented while others are underrepresented.

A component of such disproportionality is due to the fact that a seat cannot be divided, and hence it is impossible to assign the exact proportion of votes in seats terms to each party. In real practice, different methods of proportional apportionment have been proposed: D'Hondt and Sainte-Laguë (a.k.a. Jefferson and Webster, respectively, especially in the Anglo-Saxon culture), Largest Remainder (also called Hamilton), etc., which, taking into account plausible criteria, assign a whole number of seats to each party (Balinski and Young [1]; Pukelsheim [2]). Nonetheless, these methods cannot prevent the slight but not negligible¹ difference between the percentages of votes and seats that can be considered as an unavoidable structural disproportionality.

In addition to the apportionment method there are other elements of the electoral systems that may cause discordances between vote and seat percentages in elections. Mainly:

- The existence of several electoral constituencies, especially when there are many small and medium sized ones.
- The exclusion of some parties by means of electoral thresholds.
- The addition of a number of seats (bonus) assigned to the winning party.

Additional factors beyond the allocation formula influencing disproportionality are considered in Gallagher [3] and Suojanen [4], among others.

There is no unanimity on how to measure the distortions caused by translating votes into seats, and many efforts have been made to quantify such deviation. Disproportionality indexes are generally used to this end, and there is a wide literature on this subject. So, Karpov [5] compiled nineteen indexes resulting from the application of different techniques. This author also includes an interesting analysis of the properties that the indexes fulfil. Furthermore, Taagapera and Grofman [6] have made an overview of several indexes that estimate electoral disproportionality, and analyse their suitability (see also Taagapera [7]). Finally, Koppel and Diskin [8] and Boyssou et al. [9] propose characterizations for some indexes that measure disproportionality.

As far as we know, the indexes proposed in the literature use as reference the exact (non integer) number of seats that every party should receive in pure proportionality. Instead of this, in the present paper, we measure electoral results regarding feasible values, that is: The seats allotment obtained using a pre-established method of proportional representation applied to the parties' total votes without distortion elements afflicting the results (that is, as if it were a unique constituency, no electoral thresholds nor incentives to the winning party). In this way, we quantify the discrepancies between the actual allotment and that obtained using the considered apportionment method.

It is worth to note that Gallagher [3] foresaw this approach pointing out that "it would be fairer to measure disproportionality not as the difference between the actual outcome and perfect proportionality but as the difference between the actual outcome and the highest degree of proportionality that was attainable under the circumstances".

This paper is structured as follows. In the second section we briefly describe the proportional representation methods and, for them, we detail desirable properties. As a consequence, we point out as suitable procedures those of Jefferson and Webster. In the third section we survey some classic indexes of disproportionality, as well as the reasonable properties that any index should verify. In section four we

¹ By "not negligible" we mean that these differences between votes and seats shares due to rounding to whole numbers are small, but not zero. As we will show, classical disproportionality indexes take into account these differences, but our approach does not.

introduce the disproportionality index associated with a proportional representation method. We highlight in particular the indexes associated with Jefferson and Webster methods because of its properties². Section five includes comparisons of the proposed indexes with those of Gallagher, Loosemore-Hanby and Sainte-Laguë (defined in Section 3) for several elections to German Bundestag, Swedish Riksdag, Spanish Congreso de los Diputados, Portugal Assembleia da República and British House of Commons. All the electoral databases used appear in Appendix A. Finally, we present some conclusions.

2. Proportional representation

The aim in this paper is to provide a new perspective for analysing electoral disproportionality by means of a new family of indexes associated with Proportional Representation (PR) methods. This is the reason why, in what follows, we present the main PR methods and their properties. A complete description of them can be found in Balinski and Young [1] and Pukelsheim [2].

2.1 Notation

Let $\bar{V} = (V_1, V_2, \dots, V_n)$ be the total number of votes cast for the n political parties running an election and $\bar{S} = (S_1, S_2, \dots, S_n)$ be the seats that the electoral system has assigned to the parties. $V = \sum_{i=1}^n V_i$ is the total number of valid votes obtained for the different candidacies and $S = \sum_{i=1}^n S_i$ is the size of the parliament.

We denote by $q_i = \frac{V_i}{V} S$ the quota of party i , that is, the number of seats this party should receive in pure proportionality. Generally, this number is not integer; its downward rounding is called the lower quota and its upward rounding, the upper quota.

Finally, the fraction of the total votes received by the party i is given by $v_i = V_i/V$ and the fraction of seats of party i is $s_i = S_i/S$.

2.2 Proportional Representation (PR) methods

A PR problem consists in a pair (\bar{V}, S) where the components of \bar{V} are the votes obtained by the different parties and S is the number of seats to be allotted. One solution of the PR problem is given by the integer numbers of seats for every party, S_1, S_2, \dots, S_n , so that every S_i is near q_i and also $S_1 + S_2 + \dots + S_n = S$. In order to calculate these solutions, two major families of procedures have been proposed: quotient and remainder methods, and divisor methods.

First methods require to multiply \bar{V} by a scalar $k > 0$ and to allocate the integer part of kV_i to every party i . After this, one additional seat is given to the parties with the largest remainders (fractional part) in the product kV_i , up to sum the S seats. The most significant method based on quotients is the Largest Remainder Method (or Hamilton), which uses as a factor the value $k = S/V$, and hence the quantity SV_i/V is q_i , the quota of party i .

On the other hand, divisor methods establish a way to round the fractions in every interval limited by two consecutive integer numbers, that is, they establish in each interval a threshold or signpost so that any fraction below that signpost is rounded downwards, any fraction above the signpost is rounded upwards, and any fraction that coincide with the signpost allows both types of rounding (tie situation).

Once the signposts are established, in order to apply the divisor method, we have to find a value for k so that the roundings of kq_i add up to S , the total number of seats to be assigned. Divisor method

² We use the Anglo-Saxon denomination of these methods in the design of the disproportionality indexes introduced along this paper in order to avoid misunderstandings with already existing indexes (see footnote 8 for more terminology details).

allotments can be carried out, equivalently, generating a table with the quotients of the party votes divided by the established signposts. The S greatest quotients indicate the allotted seats to the corresponding parties. We describe some of the divisor methods:

1) *Jefferson (D'Hondt) method*

The signpost for the $[0, 1]$ interval is 1, for the interval $[1, 2]$ is 2, and so on. That is, the signposts are: 1, 2, 3, 4, 5, ... So, the Jefferson method assigns to the quotient kq_i its integer part.

2) *Webster (Sainte-Laguë) method*

In this case, the signposts are the centre of the intervals, that is: 0.5; 1.5; 2.5; 3.5; ... So, each one of the fractions kq_i are assigned to the nearest integer.

3) *Adams method*

For this method the signposts are: 0, 1, 2, 3, ... That is, each fraction kq_i is assigned to the rounded up integer. Because of the first signpost, parties with just only one vote could obtain representation. Hence, the Adams method is not suitable for allotting seats to political parties, unless there are restrictions to the parties to get into the allotment. This is the reason why this method is not used in real practice³.

2.3 Analysing properties of PR methods

One way to choose one PR method among all the theoretically possible is to consider properties that the methods satisfy and select the method that fulfils those deemed more relevant. Below we analyse the most significant properties (see again on this issue Balinski and Young [1], Pukelsheim [2], and Palomares et al. [10], as well as Niemeyer and Niemeyer [11]).

- i) *Exactness*. If the quotas of all the parties are integer numbers, then the allotted seats to every party should be those quotas. This is an unquestionable property, and in fact, it is verified by all the aforementioned proportional representation methods.
- ii) *House Size Monotonicity*. If the number of seats to be allotted increases, then no party should receive less seats than in the initial situation. The opposite behaviour is known as the *Alabama Paradox*. All the divisor methods are house size monotone. The Largest Remainder method is not.
- iii) *Quota*. A method satisfies the quota property if no party obtains neither more seats than its rounded up quota, nor less seats than its rounded down quota. No divisor method verifies the quota property. Largest Remainder method does verify the quota property.
- iv) *Lower Quota*. A method fulfils the lower quota property if no party receives less seats than the integer part of its quota. The only divisor method that fulfils this property is Jefferson (D'Hondt).
- v) *Near Quota*. A method is 'near quota' if for any pair of parties it is impossible to take a seat of one party and give it to the other and simultaneously bring both of them nearer their quotas. The only divisor method that fulfils this property is Webster (Sainte-Laguë).
- vi) *Schisms Penalization*. A PR method penalizes schisms, or equivalently, favours coalitions if, after a party splits into two, with the sum of the two parties votes being equal to the votes of the original party, and supposing the votes of the other parties remain unchanged, then the method assigns to the two new parties a number of seats equal to or less than the number of seats that the original party would have obtained. The only divisor method that satisfies this property is Jefferson.
- vii) *Unbiasedness*. A method is unbiased if it does not systematically favour the most voted parties over the less voted, nor vice versa. Largest Remainder method and Webster are unbiased.
- viii) *Coherence*. Once an apportionment has been undertaken, if the total number of seats obtained by a subset of political parties is reassigned among them using the same method, it is reasonable that they will receive the same number of seats. A method that guarantees this result, that is, that any part of a

³ However, the Adams method can be very useful for distributing parliament seats among the circumscriptions in proportion to their populations, because it guarantees representation to each of them, no matter how little population they have.

proportional allotment is also proportional with such method, is said coherent or consistent. The only consistent PR methods are the divisor methods.

2.4 Choosing suitable PR methods

Unfortunately, as shown above, all desirable properties are not simultaneously satisfied by any method, the Balinski-Young impossibility theorem being the formal expression of this assert. We have to drop some properties when choosing a PR method. Also notice that we cannot give the same relevance to all properties.

Regarding the quota property, this demands very little in terms of proportionality to the less voted parties, and demands a lot to the most voted parties. For example, a party with 0.5 quota may lose its representation being assigned 0 seats, or it may obtain one seat, duplicating its quota; while another party with 7.5 quota will never obtain double its quota, that is 15 seats, but instead 8 at most.

The lower quota property is interesting because its fulfilment guarantees that each party would never receive fewer seats than its rounded down quota.

Regarding house size monotonicity, it is important to avoid the Alabama Paradox, but it is even more important to require consistency. Unlike the Alabama Paradox, inconsistency is a bad behaviour that in many cases can be directly perceived by the voters (on the theoretical importance of consistency, see Palomares et al. [10]).

It is also important to prevent high fragmentations of parliaments for the sake of governability, and consequently a method that encourages coalitions would be suitable. This behaviour is particularly interesting when the districts size is large and there are no electoral thresholds. Otherwise, an excessive atomization of the political parties' spectrum with parliamentary representation may appear.

Consequently, Jefferson becomes an appropriate PR method to assign seats to the parties, at least when electoral constituencies are big and there is no electoral thresholds. This method being house size monotone and consistent, is the only PR method that favours coalitions and guarantees to every party its lower quota. Moreover, it is the only divisor method that guarantees absolute majority of seats for absolute majority of votes, when the total number of seats is odd (Palomares and Ramírez [12]). These properties support the Jefferson method as one of the most used in real practice (Gallagher [3]; Karpov [13]; Nohlen [14]).

On the other hand, as a divisor method, Webster shares with Jefferson good features as house size monotonicity and consistency. Instead of lower quota, Webster fulfils the 'near quota' property, which is a very interesting transfer condition similar to Pareto optimality in economics, (as pointed out by Balinski and Young [1]). Even more, according to these authors, "of all divisor methods, Webster's is the least likely to violate quota". These reasons, jointly with unbiasedness, a relevant property, also make Sainte-Laguë (Webster) an advisable method for impartiality purposes.

3. Classic disproportionality indexes

In this section we describe some disproportionality indexes already appearing in the literature. We focus on those most used in real practice. Next, we present some desirable properties that such indexes could verify.

3.1 Formal expressions

In order to measure the disproportionality of an electoral outcome, more than twenty indexes have been proposed in the literature (Taagapera and Grofman [6]; Karpov [5]). All of them consider that there is disproportionality if, for some party, its fraction of votes does not coincide with the corresponding

fraction of seats, that is if $s_i \neq v_i$ for some i . In that case, party i is overrepresented when $s_i > v_i$, and underrepresented when $s_i < v_i$.

None of the disproportionality indexes presented satisfies all the desirable properties (see subsection 3.2 below), and so “the perfect index” does not exist. But some of them, like Loosemore-Hanby and Gallagher, are more widely used.

The *Loosemore-Hanby* index (Loosemore and Hanby [15]), I_{LH} , is defined as

$$I_{LH} = \frac{1}{2} \sum_{i=1}^n |v_i - s_i| \quad (1)$$

It is divided by two. This is due to the fact that the seats over apportioned to some parties are compensated with the seats of other parties that received fewer seats than they deserve and we do not want to count these discrepancies twice. A variant of this index, due to Grofman, consists in changing the 2 divisor. Instead of dividing by two, it is divided by the effective number of parties⁴ E , where

$$E = \frac{1}{\sum_{i=1}^n v_i^2} \quad (\text{see Karpov [5]}). \quad (2)$$

The I_{LH} index has a very clear meaning: it is the share of seats we would have to transfer from some parties to others in order to obtain the perfect proportionality ($s_i = v_i$ for all i). Nevertheless, in practice, such transfer is not possible because of the indivisible nature of seats.

Sometimes there is only one party j having its fraction of seats greater than its fraction of votes, $s_j > v_j$. In that case, the I_{LH} index takes the same value as the maximum deviation index, defined by

$$I_{MD} = \max_{i=1, \dots, n} |v_i - s_i| \quad (3)$$

If, instead considering only the maximum deviation, we take half of the sum of the two bigger deviations, we obtain the so called Lijphart index (see Karpov [5]).

The Gallagher index (Gallagher [3]), I_G , is defined by:

$$I_G = \sqrt{\frac{1}{2} \sum_{i=1}^n (v_i - s_i)^2} \quad (4)$$

This mathematical expression tries to reduce the effects of the small differences between votes and seats ratios. This is the reason why many authors defend its use. As a drawback, this index lacks an interpretation in terms of seats transfer.

There is a variant of this index proposed by Monroe [17], defined as

$$\sqrt{\frac{\sum_{i=1}^n (v_i - s_i)^2}{1 + \sum_{i=1}^n v_i^2}} \quad (5)$$

where the denominator decreases as the number of parties increases.

Gallagher [3] defined another index, called Sainte-Laguë (I_{SL}) which is also analyzed by Goldenberg and Fisher [18], whose expression is:

$$I_{SL} = \sum_{i=1}^n v_i \left(\frac{s_i}{v_i} - 1 \right)^2 \quad (6)$$

⁴ Introduced by Laakso and Taagapera [16].

We next exemplify the use of I_{LH} and I_G . Table 1 includes votes and seats, in relative and absolute terms, of all parties in 1998 Danish elections, whose data have been obtained from [1].

Table 1. Votes/seats for parties in 1998 Danish elections

Parties	Votes	v_i	Seats	s_i
Social Democratic Party	1,223,620	0.3593	63	0.3600
Libearal Party of Denmark	817,894	0.2401	42	0.2400
Conservative People's Party	303,965	0.0892	16	0.0913
Socialist People's Party	257,406	0.0756	13	0.0743
Danish People's party	252,429	0.0741	13	0.0743
Center Democrats	146,802	0.0431	8	0.0457
Danish Social-Liberal Party	131,254	0.0385	7	0.0400
United List-Red-Green Alliance	91,933	0.0270	5	0.0286
Christian People's Party	85,656	0.0252	4	0.0229
Progress Party	82,437	0.0242	4	0.0229
Democratic Renewal	10,768	0.0032	0	0
Independent Candidates	1,833	0.0005	0	0
Total	3,405,997	1.0000	175	1.0000

From this data we obtain $I_{LH} = 0.0175$ and $I_G = 0.0042$. According to the first value, we observe that $0.0175 \cdot 175 = 3.0625$ would be the number of seats needed to be transferred in order to achieve exact proportionality; more concretely it would be needed to transfer “a little more than 3 seats” from Social Democratic Party, Conservative People's Party, Danish People's Party, Center Democrats, Danish Social-Liberal Party and United List Red-Green Alliance parties to the remaining ones. But this is impossible in real practice, because the number of seats transferred from one party to another one must be an integer number. Only by transferring a seat from Center Democrats to Democratic Renewal the value of the first index can decrease to $I_{LH} = 0.0174$. All in all, a total of $0.0174 \cdot 175 = 3.0447$ seats cannot be transferred to achieve exact proportionality. On the other hand, the I_G value is slightly smaller⁵ than the previous one, and as previously noted this has no interpretation in terms of badly allotted seats.

3.2 Desirable properties

There are several properties that are reasonably required to any disproportionality index. According to Karpov [5], some of the most basic ones are the following.

1) *Anonymity*

If we permute votes (V_1, V_2, \dots, V_n) and seats (S_1, S_2, \dots, S_n) in the same manner, the value of the index must not change.

2) *Principle of transfers*

If we transfer a seat from an overrepresented party to an underrepresented one, the value of the disproportionality index should not increase.

3) *Homogeneity regarding the votes (scale invariance)*

The value of the index must remain invariant if the votes change proportionally, that is, if we substitute V by kV for any $k > 0$. Therefore, it is irrelevant if votes are expressed in units, thousands, percentages, etc.

4) *Normalization*

The index value must be between 0 and 1. Also, for any of the extreme values, there should be at least one distribution of seats that reaches such value.

⁵ In fact, Borysiuk et al. [19] proves that always $I_G \leq I_{LH}$.

The first property is satisfied for any of the classic indexes. With regard to the principle of transfers, some authors like Karpov [5] affirm that both I_{LH} and I_G verify the property, except for unrealistic situations. However, real data from the previous example (Table 1) show that such argument is not true. Note that in the aforementioned example, the overrepresented parties are *SDP*, *Left* and *Liberal*. If we transfer one seat from any of these parties to any of the underrepresented ones, then both the values of I_{LH} and I_G increase, as one can easily check. Consequently, the definitions of seats transfer or overrepresentation have to be refined, or else no classic index is going to satisfy this principle. In fact, Goldenberg and Fisher [18] consider a more restrictive definition than the classic one in order to demonstrate that I_{SL} decreases when one seat is transferred from an overrepresented party to an underrepresented one.

Furthermore, in regards to normalization, although all the aforementioned indexes verify it, the maximum value can only be reached if all the seats are assigned to a party with zero votes, something unthinkable under any kind of allotment; furthermore, achieving the zero value requires that the allotment were exact, something very improbable in real practice. As it will be shown, this is not the case for the indexes introduced in the next section.

4. New disproportionality indexes

To motivate our proposal, we come back to Table 1, which contains the real outcome obtained by the main political parties in the 1998 Danish elections. We saw that, from this data, the Loosemore–Hanby and Gallagher indexes, as well as all the classical indexes, take non-zero values. However, for this particular case, only by transferring one seat from Center Democrats to Democratic Renewal the value of the I_{LH} index can decrease from 0.0175 to 0.0174, but not to zero; and this is similar for other indices. However, the same apportionment is obtained by the Largest Remainder method, the Webster method and many other of the parametric divisor methods (Balinski and Ramírez [20]) to proportionally allocate the seats, without corrections or interferences.

That is to say, for all these methods, it is not possible to transfer seats among parties in order to obtain a better allotment. In that sense, the appearing disproportionality is unavoidable with the methods used in real practice, and we will say that there is no other disproportionality than the one forced by the fact that the assigned seats to the parties ought to be whole numbers. As a consequence, our aim is to design disproportionality indexes that measure only the non-forced or avoidable disproportionality.

Usually, when applying different PR methods to a particular apportionment problem, they give different solutions. Hence, for each PR method we can define an index for measuring the non-forced disproportionality regarding this method⁶.

The index associated with a method M will be calculated taking into account both the total seats allotted to the parties in a particular electoral process and the seats that they would be assigned by method M if it were applied just considering one electoral circumscription without thresholds or bonus. In this way, in the previous example, the index value associated with any of Webster (Sainte-Laguë), Largest Remainder (Hamilton) or many other methods, will be zero, this fact reflecting that the obtained apportionment is optimal.

4.1. Disproportionality index associated with a PR method

Let us suppose that $\bar{R} = (R_1, R_2, \dots, R_n) = M(\bar{V}, S)$ is the allotment obtained by applying the PR method M for assigning the S seats of the parliament in proportion to the total votes of the parties without interferences such as multiple circumscriptions, exclusion thresholds and/or bonus to the winner. Also, let

⁶ This idea of non-forced disproportionality already appeared (in other context) in Martínez-Panero et al. [21], a paper where the reference for a disproportionality index was not a proportional allotment method, but the fulfilment of the quota condition.

$\bar{S} = (S_1, S_2, \dots, S_n)$ be the actual allotment obtained on the elections. We say that a party i is *overrepresented regarding the PR method M* if $S_i > R_i$ and that the party is *underrepresented regarding the PR method*⁷ if $S_i < R_i$.

Let M be a fixed PR method; then the value

$$d = \frac{1}{2} \sum_{i=1}^n |R_i - S_i| \quad (7)$$

is the number of seats allotted disproportionately to the parties in the election, in a non-forced way.

Once the integer number d is calculated, the disproportionality index associated with M , denoted by I_M , is given by $I_M = d/S$ (we divide by S for the sake of normalization).

Notice then that, given a method M , there is M -disproportionality if $I_M \neq 0$ (equivalent to $R_i \neq S_i$ for some party i). Due to the exactness property of PR methods, if there is M -disproportionality, there will also be disproportionality in the classic sense, but the converse is not true (see indexes obtained in the example corresponding to Table 1).

The new index can also be interpreted in terms of seats transfer. The product $S \cdot I_M$ is the integer number d , and this is the amount of seats to be transferred among parties to replicate the allotment which would be obtained by strictly applying the method M , without any kind of distortion.

For practical purposes, a drawback at the time of selecting an index associated with a PR method consists in choosing a particular one among the variety of possibilities. Nonetheless, the number of interesting PR methods is small, and thus the same happens with the disproportionality indexes associated with a method. In real practice, due to the suitable properties of the Jefferson (D'Hondt) method and Webster (Sainte-Laguë) method (see subsection 2.4), in this paper we consider the indexes associated with both of them⁸, I_J and I_W , to measure disproportionality.

For illustrating our proposal, in Table 2 we calculate the value of I_J and I_W for the 2013 German elections (see [II] for sources). The electoral disproportionality in that election was the greatest in that country for the last 40 years because the *FDP* and *AfD* parties did not participate in the allotment, due to they stayed some decimals below the 5% electoral threshold. In total, the parties that obtained no representation added up near 7 million votes, which is equivalent to near the 16% of the total.

The fourth and sixth columns of Table 2 includes the 631 seats (which was the size of the Bundestag on that term) assigned under the Jefferson and Webster methods, respectively, using proportionality to the total parties' votes without considering any other restriction. The third column includes the results obtained by the German electoral system (considering the 5% barrier). These data allow us to calculate the value of I_J and I_W .

The Jefferson method would assign 94 seats to the parties that did not reach the 5% threshold. Such parties range from *FDP*, which would get 30 seats, to *pro Deutschland*, which would be assigned one seat. All of them were underrepresented in 2013 with regard the Jefferson method. Those 94 seats were assigned by the German electoral system to the five most voted parties, which were the overrepresented ones. So, $I_J = \frac{94}{631} = 0.14897$; in other words, 14.897% of the seats should be transferred to obtain the Jefferson allotment without distortions. Similarly, the value of I_W is $I_W = \frac{96}{631} = 0.1521$ (slightly more than I_J).

⁷ In what follows, we use just "over/underrepresented" referring to the classic context of disproportionality (see Subsection 3.1), and " M -over/underrepresented" when dealing with this new approach of M -disproportionality.

⁸ In the literature the terms of D'Hondt and Sainte-Laguë indexes are already coined, with a distinct meaning to that considered here (see, for example, Karpov [5]; Goldenberg and Fisher 2017). This reason why, from now on, we refer to Jefferson and Webster indexes as the ones *associated with* the Jefferson-D'Hondt and Webster-Sainte-Laguë methods, respectively.

Table 2. 2013 Bundestag election.

Parties	\bar{V} =Votes	\bar{S} =Seats	\bar{R}_j	$ \bar{R}_j - \bar{S} $	\bar{R}_w	$ \bar{R}_w - \bar{S} $
CDU	14,921,877	255	218	37	217	38
SPD	11,252,215	193	164	29	163	30
DIE LINKE	3,755,699	64	54	10	54	10
GRÜNE	3,694,057	63	54	9	54	9
CSU	3,243,569	56	47	9	47	9
FDP	2,083,533	0	30	30	30	30
AfD	2,056,985	0	30	30	30	30
PIRATEN	959,177	0	14	14	14	14
NPD	560,828	0	8	8	8	8
FREIE WÄHLER	423,977	0	6	6	6	6
Tierschutzpartei	140,366	0	2	2	2	2
ÖDP	127,088	0	1	1	2	2
REP	91,193	0	1	1	1	1
Die PARTEI	78,674	0	1	1	1	1
pro Deutschland	73,854	0	1	1	1	1
BP	57,395	0	0	0	1	1
Volksabstimmung	28,654	0	0	0	0	0
RENTNER	25,134	0	0	0	0	0
Rest (12 parties)	152,581	0	0	0	0	0
Total	43,726,856	631	631	188	631	192

For this election, $I_{LH} = 0.15678$ and $I_G = 0.0783$; thus the value of the first index doubles the second one. The I_{LH} value means that the 15.678% of the seats have to be transferred to achieve the exact proportionality. Notice that, in this example, I_{LH} is very similar to I_j and I_w , while the value of the Gallagher index is far from the other three indexes.

Likewise, in order to show here other comparisons, the maximum deviation index is $I_{MD} = 0.0628$ something lower to the Gallagher index; an even lower is the Lijphart index which equals 0.0557. A similar computation to the previous ones is given by the Grofman index, which is 0.0652. Hence, those indexes of Gallagher, maximum deviation, Lijphart and Grofman give values which raise no suspicions that two parties (*FDP* and *AfD*), gathering almost the 10% of the total votes, obtained no representation. In this case, the Monroe index is 0.1007 and it reflects something better than the four previous indexes the disproportionality of this election; and $I_{SL} = 0.1861$, which is equivalent to three percentage points more than the exact proportionality required by I_{LH} .

4.2 Interpretation and properties of the index associated with a method M

- The I_M value is easy to calculate, and has a clear and transparent meaning, as it represents the fraction of seats that have to be transferred to cancel the non-forced disproportionality regarding such method.

- The value of I_M is zero whenever there is only one electoral circumscription, the PR method used is M , and there is no other requirements for the parties to enter the seats allotment⁹. There is also the possibility to reach the zero value with more than one electoral circumscription¹⁰.
- The value of I_M is not affected by the existence of parties which obtain no seats in an election because they do not have enough votes (insufficient to obtain a seat when the M method is applied nationwide).
- If we transfer a seat from an M -overrepresented party to another M -underrepresented party, the d value decreases one unit, and the value of I_M decreases $\frac{1}{S}$.
- I_M could take any value in the set $\{0, \frac{1}{S}, \frac{2}{S}, \dots, \frac{S-1}{S}, 1\}$. Besides, while the classical disproportionality indexes only reach value 1 in situations in which seats are assigned to parties with zero votes, I_M equals 1 in more realistic contexts: For example, if all the circumscriptions are single-seat constituencies and in all of them the seats are won by a local party with insufficient votes to get it if the allotment were undertaken nationwide.
- Evidently, I_M is anonymous and homogeneous in regards to the votes.

5. Empirical application to different elections and countries

Due to the reasons argued in Subsection 2.4, we are now going to select as reference methods those of Jefferson and Webster in order to compare the behaviour of their associated disproportionality indexes in contrast with Gallagher, Loosemore-Hanby and Sainte-Laguë ones for some electoral outcomes in five countries with very different electoral systems: Germany, Sweden, Spain, Portugal and United Kingdom. We notice that the Adams method is interesting in order to assign seats to the constituencies, but not to political parties without restrictions, and this is the reason why it is not considered here.

- Germany uses a mixed member electoral system. 299 seats are elected in single-member constituencies; but in addition, at least the same number of seats is assigned to the parties exceeding a 5% threshold proportionally to their number of votes. This considerably corrects the disproportionality coming from the majoritarian electoral system.
- Sweden elects 310 seats on 29 multi-nominal circumscriptions of different sizes which create some disproportionality but, then, other 39 so-called compensatory seats are used to correct this and to achieve high proportionality among the parties exceeding the 4% nationwide. The same procedure is used in other Baltic countries.
- Spain has a Parliament with 350 seats (one more than Sweden) elected in 52 circumscriptions of different sizes, being the medium sized ones much less populated than in Sweden. An exclusion threshold of 3% in every circumscription is also considered. The size of the constituencies can cause a significant disproportionality and there are no seats to correct this behaviour like in Germany or Sweden.
- Portugal elects 230 seats in 22 constituencies. The deputies are chosen in each constituency using the Jefferson method with no legal electoral threshold. As in Spain, there are no provisions to correct the global disproportionalities caused by the 22 independent elections.

⁹ Usually an electoral system includes more than one electoral circumscription and in many occasions also an electoral threshold, to prevent the less voted parties to enter in the allotment. Even in some cases, as in Greece, the electoral system sets aside a certain number of seats to give a bonus to the winner party (Bedock and Sauger [22]). In these circumstances, if the allotment is carried out by method M , the index associated with this method shows the non-forced or unavoidable disproportionality due to such interferences.

¹⁰ It is possible to achieve a zero value for I_M even when there are several electoral circumscriptions with pre-established sizes, because we can use method M to assign seats (R_1, R_2, \dots, R_n) to the parties, and afterwards a biproportional apportionment (Balinski and Pukelsheim [23]; Ramírez et al. [24]) to distribute the seats to the different circumscriptions.

- United Kingdom chooses the 650 deputies of the House of Commons using a majority system, which generates a very high disproportionality.

Thus, the five selected countries illustrate a very wide spectrum of electoral systems. Besides, in every one of them we will calculate the disproportionality indexes values for the outcomes of several elections, 55 altogether.

5.1 Application to the German Bundestag

Table 3 shows the percentage values of I_G , I_J , I_W , I_{LH} and I_{SL} for the German Bundestag elections held from 1976 to the most recent in 2017. Notice that the size of this parliament (S) varies along electoral calls. The following data have been obtained from [II].

Table 3. Disproportionality indexes. Germany elections 1976-2017.

Election	I_G	I_J	I_W	I_{LH}	I_{SL}	S
2017	1.95	3.67	4.80	5.00	5.26	709
2013	7.83	14.90	15.21	15.69	18.61	631
2009	3.40	4.66	5.95	6.01	6.68	622
2005	2.28	3.09	3.75	4.33	4.56	614
2002	3.83	5.97	6.47	6.70	6.89	672
1998	2.75	3.59	4.33	4.72	5.00	669
1994	2.22	2.83	3.42	3.61	3.78	672
1990	4.62	7.40	8.01	8.05	8.78	662
1987	0.71	0.80	1.01	1.35	1.38	497
1983	0.50	0.20	0.80	0.79	0.53	498
1980	1.41	1.41	1.81	1.98	2.02	497
1976	0.57	0.40	0.81	0.94	0.89	496
Mean	2.67	4.08	4.70	4.93	5.37	

The 2013 data are an exception (parties under the electoral threshold obtained approximately 16% of the votes), and negatively affect the means in the last row. As we noted in the previous section, the values of I_J , I_W and I_{LH} reflect this situation better than the Gallagher index. Notice that, with the exception of 1983 elections, I_{SL} reaches the highest value among the five considered indexes.

In 1990 The Greens and The Republicans obtained no representation, adding together the 5.9% of the votes. In 2002, the *SPD* got 4% of the votes, and they also were left out the parliament. In these two occasions the values of the indexes show how disproportionality has increased, but much less than in 2013. In almost all the elections the values of the five indexes have followed the ordering $I_G < I_J < I_W < I_{LH} < I_{SL}$, unless when the disproportionality is very low. In such cases, I_J is usually nearer the Gallagher index; even twice (1976 and 1983) it turns out to become lower than I_G .

5.2 Application to the Swedish Riskdag

Table 4 shows the values of I_G , I_J , I_W , I_{LH} and I_{SL} for all the Riskdag elections held in Sweden from 1998 to 2014. In [III] there are the electoral outcomes regarding the elections.

Table 4. Disproportionality indexes. Sweden elections 1998-2014 (S=349)

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2014	2.64	3.44	3.44	4.04	4.26
2010	1.25	1.43	2.01	2.07	1.49
2006	3.31	4.58	6.30	6.87	7.39
2002	1.55	2.01	2.58	2.85	2.93
1998	1.27	1.43	2.58	2.52	2.59
Mean	2.04	2.58	3.38	3.67	3.73

As we note above, the Swedish electoral system causes a high proportionality unless one or several political parties had a vote percentage somewhat lower the electoral threshold, which currently is 4%. That happened in 2014 as the *Feminist Initiative* party got the 3.1% of the votes, and also in 2006 because *Sweden Democrats* got 2.9%. In this case, the ordering $I_G < I_J < I_W < I_{LH} < I_{SL}$ is satisfied for all elections except that of 2010, the most proportional one. Then, I_{SL} reached the lowest value, while all the other indexes kept their relationships.

5.3 Application to the Spanish Congreso de los Diputados

Table 5 contains the values of I_G, I_J, I_W, I_{LH} and I_{SL} in all the parliamentary elections held in Spain from 1977 to 2019a (April). In this case the electoral data have been taken from [IV].

Table 5. Disproportionality indexes. Spanish elections 1977-2019a (S=350)

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2019a	5.52	8.57	8.57	9.64	6.98
2016	5.25	6.29	7.71	7.85	5.04
2015	5.94	9.43	10.29	10.54	8.71
2011	6.92	9.14	10.57	11.30	9.92
2008	4.51	4.86	7.14	8.09	7.58
2004	4.63	4.86	7.43	7.96	6.79
2000	5.61	6.00	7.43	8.59	7.54
1996	5.33	5.71	7.43	8.08	5.69
1993	6.82	9.43	11.14	12.01	10.61
1989	8.97	12.29	14.29	15.10	13.72
1986	7.35	10.00	12.00	12.69	11.28
1982	8.17	10.57	12.86	13.87	12.67
1979	10.56	14.57	16.86	17.66	17.94
1977	10.40	14.57	16.29	18.14	18.06
Mean	6.86	9.02	10.71	11.54	10.18

As we can see, with all the indexes that we are considering, the Spanish electoral system has a significantly higher disproportionality than the German and Swedish ones. This happens because in Spain

the Jefferson-D'Hondt method is used to assign the seats, and because there are 52 circumscriptions, mainly of medium or small size. We notice again that all four first indexes keep the same ordering in every election: $I_G < I_J < I_W < I_{LH}$. However, I_{SL} ranges all positions, from staying below Gallagher in 2016 to being over I_{LH} in 1979.

5.4 Application to Assembleia da República Portuguesa.

Table 6 contains the values of I_G, I_J, I_W, I_{LH} and I_{SL} indexes in every parliamentary elections held in Portugal from 1975 to 2015. Electoral data have been obtained from [V].

Portugal has an electoral system with closed and blocked lists, and uses Jefferson-D'Hondt method to carry out the apportionment, as in Spain, but the mean size of the circumscription is greater than 10 seats, while the mean size in Spain is 7 seats. Moreover, the three biggest circumscriptions in Portugal constitute more than half of the Assembleia seats. This largely justifies the higher proportionality of the electoral outcomes in comparison with that of Spain. Important deviations arise due to the most voted parties. For example, in each of the four last elections, among both of them have received about 21 seats of surplus (more than 9% of all 226 seats), and such disproportionality is far from what I_G and I_{SL} measure, even somehow far from I_J .

Table 6. Disproportionality. Portugal elections 1975-2015

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2015	5.41	7.52	10.62	10.20	8.76
2011	5.42	6.64	9.29	9.29	6.93
2009	5.41	7.52	8.85	9.27	5.89
2005	5.82	7.52	9.29	8.98	5.68
2002	4.65	6.19	7.96	7.64	4.42
1999	4.34	5.31	7.08	7.09	4.32
1995	4.54	6.19	6.64	7.38	4.57
1991	5.60	7.08	7.96	8.55	6.88
1987	6.13	7.32	8.54	8.97	7.91
1985	3.63	4.87	6.10	5.76	4.49
1983	2.97	4.07	4.88	5.22	4.01
1980	3.96	4.47	6.10	5.98	5.60
1979	3.78	4.87	6.10	5.91	5.68
1976	3.70	4.63	6.56	6.50	6.05
1975	5.71	8.91	10.12	9.91	7.47
Mean	4.74	6.21	7.74	7.78	5.91

5.5 Application to the House of Commons in United Kingdom

Finally, Table 7 shows the behaviour of the five indexes in elections which use a majoritarian system; specifically they are computed for all the United Kingdom House of Commons elections from 1983, whose data have been obtained from [VI].

Table 7. Disproportionality. United Kingdom elections 1983-2015 (S=650)

Election	I_G	I_J	I_W	I_{LH}	I_{SL}
2017	6.36	9.55	9.71	9.68	11.45
2015	14.92	23.11	23.42	23.38	31.55
2010	14.94	21.57	21.88	21.87	23.31
2005	16.57	20.16	20.31	20.14	23.06
2001	17.44	21.09	21.09	21.10	20.70
1997	16.30	20.52	20.52	20.56	20.47
1992	13.33	16.59	16.74	16.74	17.70
1987	17.68	20.15	20.31	20.30	23.70
1983	20.50	23.23	23.54	23.48	29.26
Mean	15.34	19.55	19.72	19.69	22.36

As could be expected, due to the majoritarian nature of its electoral system, the disproportionality in United Kingdom is much higher than in the previously considered countries. It affects near 20% of the seats in almost all the elections.

Note that here the value of I_J and I_W are almost identical to that of I_{LH} in all the elections. However, I_G always stays several points below them, and, with few exceptions, I_{SL} usually reaches the highest value, sometimes exaggeratedly, as it happened in 2015, taking a value of 0.3155 while I_{LH} was 0.2338.

5.6 Discussion

- 1) The indexes I_J , I_W and I_{LH} have a very clear interpretation and their computation is very simple. In addition, the calculation of I_J and I_W does not require to take into account the parties with small quota (significantly less than 1 and 0.5, respectively), if they have not received seats in the corresponding election.
- 2) The value of I_G is usually lower than those of I_J , I_W and I_{LH} , and does not have a specific meaning. In most cases, but not always, these four indexes are ordered as follows $I_G < I_J < I_W < I_{LH}$.
- 3) The value of I_{SL} is less sensitive to differences in representation of large parties rather than in medium and less voted ones; that is because it uses the quotient between fraction of votes and seats. Its value does not inform us if we are far or near the exact proportionality. From its value of 0.3155 in 2015 UK election, we cannot imagine that the disproportionality was then below 25%, nor from the value of 0.0568 in 2005 Portugal election we can know that the disproportionality was around 9%. In addition its value depends on the parties' size with disproportionality in the representation. Therefore, in some countries it is below I_G and in others it is the one that takes the highest value.

Table 8 contains, in a comprehensive way, the average values of the considered indexes in the analysed elections for each country.

Table 8. Indexes averages in the five countries

Country	I_G	I_J	I_W	I_{LH}	I_{SL}
Germany	2.67	4.08	4.70	4.93	5.37
Sweden	2.04	2.58	3.38	3.67	3.73
Spain	6.86	9.02	10.71	11.54	10.18
Portugal	4.74	6.21	7.74	7.78	5.91
UK	15.34	19.55	19.72	19.69	22.36

6. Conclusions

In this paper we consider a new scheme to measure electoral disproportionality. Contrary to the classic approach, which takes as reference the unattainable exact proportionality, our proposal takes into account deviations from feasible values: The seats distributions obtained when a prefixed proportional allotment method is carried out without interferences such as the existence of several circumscriptions, thresholds of exclusion and/or bonus for the winner. In this way, we quantify the discrepancies between the actual seat allotment and that obtained by the considered method.

The indexes associated with a method, proposed in this work, are easy to calculate and their values have a very clear meaning. When they are multiplied by the size of the parliament, they show the number of seats needed to be transferred, from M -overrepresented to M -underrepresented parties, in order to get a proportional allotment with regard to the chosen method. Thus, these indexes quantify the distortions produced by electoral thresholds, circumscriptions and direct bonus to the winning party. However, they do not measure the forced disproportionality due to the fact that the seats allotment has to consist of whole numbers.

It is important to note that it is possible to design an electoral system with zero disproportionality, in the manner introduced in this paper, even maintaining several circumscriptions. This can be accomplished if the seats are assigned to the parties in proportion to the total votes, and then, those seats are distributed to the circumscriptions using a biproportional method, so that every electoral circumscription will get the previously established seats.

Due to the positive properties of the Jefferson (D'Hondt) and Webster (Sainte-Laguë) methods, we advocate for the suitability of the indexes associated with these methods (I_J and I_W). We have contrasted such indexes with classic ones: Gallagher, Loosemore-Hanby and Sainte-Laguë, (I_G , I_{LH} and I_{SL}), in 55 electoral processes held in the last decades in countries with very different electoral systems: Germany, Sweden, Spain, Portugal and United Kingdom. In most of the cases $I_G < I_J < I_W < I_{LH}$, being I_W and I_{LH} near values, while I_J is closer to I_W than I_G . However, the relationship among I_{SL} and the other indexes varies: 23 times reaches the highest position, but usually reaches intermediate positions and sometimes stays below I_G . This is because the same difference between votes and seats is reflected in this index in a different way depending of the size of the party.

From the arguments exposed along this paper, it is possible to conclude that the indexes associated with the Jefferson and Webster methods are valid alternatives to the hitherto considered indexes to measure the disproportionality of electoral outcomes.

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Appendix A. Electoral databases

I. Denmark

<http://electionresources.org/dk/>

II. Germany

<https://www.bundeswahlleiter.de/bundestagswahlen/2017/publikationen.html>

III. Sweden

<http://electionresources.org/se/>

<https://data.val.se/val/val2018/slutresultat/R/rike/index.html>

IV. Spain

<http://www.infoelectoral.mir.es/infoelectoral/min/>

V. Portugal

<http://electionresources.org/pt/>

VI. United Kingdom

<http://electionresources.org/uk/>